

# Discounted optimal stopping zero-sum games in diffusion-type models with maxima and minima

Pavel V. Gapeev\*

We present closed-form solutions to a discounted optimal stopping zero-sum game in a model with a generalised geometric Brownian motion with coefficients depending on its running maximum and minimum processes. The optimal stopping times forming a Nash equilibrium are shown to be the first times at which the original process hits certain boundaries depending on the running values of the associated maximum and minimum processes. The proof is based on the reduction of the original game to the equivalent free-boundary problem and the solution of the latter problem by means of the smooth-fit and normal-reflection conditions. We show that the optimal stopping boundaries are partially determined as either unique solutions of the appropriate system of arithmetic equations or unique solutions of the appropriate first-order nonlinear ordinary differential equations. The obtained results are related to the valuation of the perpetual lookback game options with floating strikes in the appropriate diffusion-type extension of the Black-Merton-Scholes model.

## 1. Formulation of the problem

The main aim of this paper is to present closed-form solutions to a discounted optimal stopping zero-sum game with the value:

$$V_* = \sup_{\tau} \inf_{\zeta} E \left[ e^{-r\tau} (S_{\tau} - K X_{\tau}) I(\tau < \zeta) + e^{-r\zeta} (Q_{\zeta} - L X_{\zeta}) I(\zeta \leq \tau) \right] \quad (1.1)$$

for some given constants  $0 < L < K \leq 1$ , where  $I(\cdot)$  denotes the indicator function. Here, for a precise formulation of the problem, let us consider a probability space  $(\Omega, \mathcal{F}, P)$  with a standard Brownian motion  $B = (B_t)_{t \geq 0}$ . Suppose that the process  $X = (X_t)_{t \geq 0}$  is given by:

$$X_t = x \exp \left( \int_0^t \left( r - \delta(S_u, Q_u) - \frac{\sigma^2(S_u, Q_u)}{2} \right) du + \int_0^t \sigma(S_u, Q_u) dB_u \right) \quad (1.2)$$

---

\*London School of Economics, Department of Mathematics, Houghton Street, London WC2A 2AE, United Kingdom; e-mail: p.v.gapeev@lse.ac.uk

*Mathematics Subject Classification 2010:* Primary 60G40, 34B40, 91G20. Secondary 60J60, 60J65, 91B70.

*Key words and phrases:* Three-dimensional optimal stopping zero-sum game, Brownian motion, running maximum and minimum processes, free-boundary problem, instantaneous stopping and smooth fit, normal reflection, a change-of-variable formula with local time on surfaces, perpetual American lookback game options.

*Date:* October 17, 2021

so that it solves the stochastic differential equation:

$$dX_t = (r - \delta(S_t, Q_t)) X_t dt + \sigma(S_t, Q_t) X_t dB_t \quad (X_0 = x) \quad (1.3)$$

where  $r > 0$  is a given constant,  $\delta(s, q) > 0$  and  $\sigma(s, q) > 0$  are continuously differentiable bounded functions on  $[0, \infty]^2$ , and  $x > 0$  is fixed. We further assume that the function  $\delta(s, q)$  is increasing in the both variables  $s$  and  $q$  on  $[0, \infty]^2$ . Here,  $S = (S_t)_{t \geq 0}$  and  $Q = (Q_t)_{t \geq 0}$  are the associated with  $X$  *running maximum* and *minimum* processes defined by:

$$S_t = s \vee \max_{0 \leq u \leq t} X_u \quad \text{and} \quad Q_t = q \wedge \min_{0 \leq u \leq t} X_u \quad (1.4)$$

for arbitrary  $0 < q \leq x \leq s$ , respectively. Since the functions  $\delta(s, q)$  and  $\sigma(s, q)$  are assumed to be bounded on  $[0, \infty]^2$ , it follows from [46; Chapter IV, Theorem 4.8] that there exists a (pathwise) unique solution to the stochastic differential equation in (1.3). It can be assumed that the process  $X$  describes the price of a risky asset on a financial market, where  $r$  is the riskless interest rate,  $\delta(s, q)$  is the dividend rate paid to the asset holders, and  $\sigma(s, q)$  is the volatility rate. Suppose that the supremum and infimum in (1.1) are taken over all stopping times  $\tau$  and  $\zeta$  of the process  $X$ , and the expectations there are taken with respect to the risk-neutral probability measure  $P$ . In that case, the value of (1.1) can therefore be interpreted as the rational (or no-arbitrage) price of a perpetual lookback game (or Israeli) option with the floating strikes  $KX$  and  $LX$  in the diffusion-type extension of the Black-Merton-Scholes model introduced in Gapeev and Rodosthenous [28]-[30]. Such game-type contingent claims were introduced by Kifer [39] and further studied by Kyprianou [43], Kühn and Kyprianou [42], Kallsen and Kühn [37], Baurdoux and Kyprianou [3]-[5], Gapeev and Kühn [23], Ekström and Villeneuve [15], Ekström and Peskir [14], Peskir [53]-[54], and Baurdoux, Kyprianou, and Pardo [6] among others. We also refer to Shiryaev [66; Chapter VIII; Section 2a], Peskir and Shiryaev [57; Chapter VII; Section 25], and Detemple [12] for extensive overviews of the solutions to the American option pricing problems as well as other related results on optimal stopping problems in financial mathematics.

Discounted optimal stopping problems for certain reward functionals depending on the running maxima and minima of continuous Markov (diffusion-type) processes were initiated by Shepp and Shiryaev [62] and further developed by Pedersen [49], Guo and Shepp [34], Peskir [51], Gapeev [19]-[20], Guo and Zervos [35], Peskir [55]-[56], Glover, Hulley, and Peskir [31], [28]-[30], Kitapbayev [40], Rodosthenous and Zervos [61], Gapeev, Kort, and Lavrutich [22], Gapeev and Li [25]-[26], and Gapeev and Al Motairi [27] among others. The main feature in the analysis of such optimal stopping problems was that the normal-reflection conditions hold for the value functions at the diagonal planes of the state spaces of the multi-dimensional continuous Markov processes having the initial processes and the running extrema as their components. It was shown, by using the established by Peskir [50] maximality principle for solutions of optimal stopping problems, which is equivalent to the superharmonic characterisation of the value functions, that the optimal stopping boundaries are characterised by the appropriate extremal solutions of certain (systems of) first-order nonlinear ordinary differential equations. In this paper, we continue this development and study the problem of (1.1) related to the pricing of the floating-strike lookback game options as the associated optimal stopping zero-sum game of (2.3) for a three-dimensional (continuous) Markov diffusion-type process which

has the underlying risky asset price  $X$  as well as its running maximum  $S$  and minimum  $Q$  as their state space components.

Note that the resulting problems turn out to be necessarily three-dimensional in the sense that they cannot be reduced to optimal stopping problems for Markov processes of lower dimensions. It is shown that the optimal exercise times forming a Nash equilibrium are the first times at which the original process exits certain two-sided regions restricted by stochastic boundaries depending on the running values of the associated maximum and minimum processes. We apply the smooth-fit and normal-reflection conditions for the value functions to determine the optimal stopping boundaries as either unique solutions of the appropriate system of arithmetic equations or unique solutions of the appropriate first-order nonlinear ordinary differential equations. Optimal stopping problems with the appropriate one-sided continuation regions in similar models based on the original diffusion-type processes with coefficients depending on the running maximum and the running maximum drawdown were considered in [28]-[30]. Other optimal stopping problems in models with spectrally negative Lévy processes and their running maxima were studied by Asmussen, Avram, and Pistorius [1], Avram, Kyprianou, and Pistorius [2], Ott [48], and Kyprianou and Ott [44] among others.

The dependence of the local drift and diffusion coefficients on the past dynamics of observable diffusion-type processes through certain processes playing the role of sufficient statistics is often used in financial practice and well studied in the related literature. For instance, an increase of the running maximum or decrease of the running minimum of a risky asset price normally causes a structural change in the local drift representing its expected return and dividend policy. It also triggers changes in the diffusion coefficient representing the volatility rate of an asset price with a higher impact under either a maximum increase or a minimum decrease rather than either a minimum increase or maximum decrease, respectively. Such sufficient statistics transparently exhibit the risk levels of the assets and therefore usually influence the decisions taken by the market participants. The demand for option pricing in models with stochastic interest rates and volatility initiated the development and subsequent calibration of these models, based on diffusion-type processes with tractable path-dependent coefficients, which were realised by Henry-Labordère [36] and Ren, Madan, and Qian [59] among others (see also [29] for further discussions on diffusion-type models for prices of financial assets with coefficients depending on the running maxima and minima as well as the maxima drawdowns and maxima drawups).

The paper is organised as follows. In Section 2, we formulate the optimal stopping zero-sum game for a necessarily three-dimensional continuous Markov process, which has the underlying asset price and the running values of its maximum and minimum as the state space components. The resulting optimal stopping game is reduced to the equivalent free-boundary problem for the value function which satisfies the smooth-fit conditions at the stopping boundaries and the normal-reflection conditions at the edges of the state space of the three-dimensional process. In Section 3, we obtain closed-form expressions for the candidate value functions as well as derive the appropriate arithmetic equations and first-order nonlinear ordinary differential equations for the candidate stopping boundaries as solutions of the associated free-boundary problems. We specify the starting conditions for the solutions of the first-order nonlinear ordinary differential equations and provide a recursive algorithm to determine the value functions and the optimal stopping boundaries along with their intersection lines with the edges of the three-dimensional state space. In Section 4, by applying the change-of-variable formula with local time on surfaces

from Peskir [52], it is verified that the resulting solution of the free-boundary problem provides the expressions for the value function and the optimal stopping boundaries for the underlying asset price process in the original problem. In Section 5, we give closed-form solutions to some auxiliary optimal stopping problems in the same model, which give the appropriate bounds for the value functions and optimal stopping boundaries for the original game. We apply the maximality principle from Peskir [50] to the framework of the three-dimensional optimal stopping problems to show that the optimal stopping boundaries provide the extremal solutions of the associated first-order nonlinear ordinary differential equations (see also [56] and [28]-[30] for other optimal stopping problems in other related three-dimensional models). The main results of the paper are stated in Theorems 4.1 and 5.1.

## 2. The optimal stopping game and free-boundary problem

In this section, we introduce the setting and notation of the three-dimensional optimal stopping zero-sum game associated with the value of (1.1), which is related to the pricing of the perpetual floating-strike lookback game options. We specify the structure of the optimal stopping times forming a Nash equilibrium and formulate the equivalent free-boundary problem.

**2.1 The three-dimensional optimal stopping zero-sum game.** Suppose that an investor writes a perpetual lookback game option and sells the contract to another investor at time 0. Then, the holder of the option can exercise the contract at some random time  $\tau$  which they can choose by collecting the amount of the running maximum  $S$  and paying the floating strike  $KX$  to the writer. At the same time, the writer of the option can recall the contract at some random time  $\zeta$  which they can choose by paying the amount of running minimum  $Q$  to the holder and collecting the floating strike  $LX$ . In this respect, the holder of the option looks for the exercise time  $\tau_*$  maximising the expected total payoff received from the writer, while, at the same time, the writer of the contract looks for the recall time  $\zeta_*$  minimising the expected total payoff sent to the holder. In other words, the perpetual lookback game option pricing problem seeks to determine the couple of stopping times  $\tau_*$  and  $\zeta_*$  of the process  $X$  being a *saddle point* for the total expected reward functional given by:

$$J(\tau, \zeta) = E[e^{-r\tau} (S_\tau - K X_\tau) I(\tau < \zeta) + e^{-r\zeta} (Q_\zeta - L X_\zeta) I(\zeta \leq \tau)] \quad (2.1)$$

which means that the inequalities:

$$J(\tau, \zeta_*) \leq J(\tau_*, \zeta_*) \leq J(\tau_*, \zeta) \quad (2.2)$$

should hold, for any exercise and recall times  $\tau$  and  $\zeta$ . Such a couple  $\tau_*$  and  $\zeta_*$  satisfying the inequalities of (2.2) with (2.1) defined above is called the *Nash equilibrium* in the optimal stopping zero-sum game of (1.1) (see, e.g. [8], [14] and [53] for a precise definition of this notion).

It thus follows from the results of [39] and [37] that the rational (or no-arbitrage) price of the game contingent claim described above coincides with the value function  $V_*(x, s, q)$  of the optimal stopping zero-sum game for the (time-homogeneous strong) Markov process  $(X, S, Q) = (X_t, S_t, Q_t)_{t \geq 0}$  of the form:

$$V_*(x, s, q) = \sup_{\tau} \inf_{\zeta} E_{x,s,q} [e^{-r\tau} (S_\tau - K X_\tau) I(\tau < \zeta) + e^{-r\zeta} (Q_\zeta - L X_\zeta) I(\zeta \leq \tau)] \quad (2.3)$$

for some given constants  $0 < L < K \leq 1$ , where the supremum and infimum are taken over all stopping times  $\tau$  and  $\zeta$  with respect to the natural filtration  $(\mathcal{F}_t)_{t \geq 0}$  of the process  $X$ . Here, we denote by  $E_{x,s,q}$  the expectation under the assumption that the three-dimensional (strong Markov) process  $(X, S, Q)$  defined in (1.2)-(1.3) and (1.4) starts at  $(x, s, q) \in E$ , and by  $E = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q \leq x \leq s\}$  the state space of the process  $(X, S, Q)$ . It follows from the results of [11; Theorem 4.1] based on the solutions of the associated (doubly) reflected backward stochastic differential equations that the game-type optimal stopping problem of (2.3) has a value. The existence of the associated *Stackelberg equilibria* in various optimal stopping games is proved in the results of [67], [45], and [53]-[54] among others. We further establish the existence and describe the structure of the stopping times  $\tau_*$  and  $\zeta_*$  forming a *Nash equilibrium* of the optimal stopping zero-sum game of (2.3).

Observe that, according to the structure of the running maximum and minimum processes  $S$  and  $Q$  in (1.4), if the inequality  $K \leq 1$  holds, then the exercise payoff  $S_t - KX_t$  is always positive, while, if the inequality  $L \geq 1$  holds, then the recall payoff  $Q_t - LX_t$  is always negative, for all  $t \geq 0$ . In this view, taking into account the property of the reward functionals of optimal stopping zero-sum games that the recall payoff should exceed the exercise payoff in this setting (see, e.g. [14] and [53] for further discussions), so that the inequality  $S_t - KX_t \leq Q_t - LX_t$  should be satisfied, for all  $t \geq 0$ , we need to inherently assume that the inequalities  $0 < L < K \leq 1$  hold. This assumption allows to keep the both payoffs positive, that also represents an important feature for callable financial contracts. Note that the latter inequality between the payoffs tells us that we should consider the points  $(x, s, q) \in E$  such that the inequality  $s - q \leq (K - L)x$  holds, which is equivalent to  $x \geq a'(s, q)$  with  $a'(s, q) = (s - q)/(K - L)$ , since the solution of the problem is degenerated otherwise (see Subsection 5.1 below for further discussions of properties of the payoffs under various conditions on the strikes  $K$  and  $L$ ).

**2.2 The structure of optimal stopping times.** Let us first determine the structure of the stopping times forming a Nash equilibrium in the optimal stopping game of (2.3).

(i) By means of the results of general optimal stopping theory for Markov processes (see, e.g. [57; Chapter I, Section 2.2]) and the results of general theory of optimal stopping games (see, e.g. [7]-[8], [16]-[17], [41], [67], [45], and [11] among others), we obtain from the structure of the reward functional that the stopping times forming a Nash equilibrium in the optimal stopping game of (2.3) exist and are given by:

$$\tau_* = \inf \{t \geq 0 \mid V_*(X_t, S_t, Q_t) = S_t - K X_t\} \quad (2.4)$$

and

$$\zeta_* = \inf \{t \geq 0 \mid V_*(X_t, S_t, Q_t) = Q_t - L X_t\} \quad (2.5)$$

so that the associated continuation and stopping regions have the form:

$$C_* = \{(x, s, q) \in E \mid s - Kx < V_*(x, s, q) < q - Lx\} \quad (2.6)$$

and

$$D_* = \{(x, s, q) \in E \mid \text{either } V_*(x, s, q) = s - Kx \text{ or } V_*(x, s, q) = q - Lx\} \quad (2.7)$$

respectively. It is seen from the results of Theorem 4.1 below that the value function  $V_*(x, s, q)$  is continuous, so that the set  $C_*$  is open and the set  $D_*$  is closed.

(ii) We now describe the structure of the continuation and stopping regions  $C_*$  and  $D_*$  from (2.6)-(2.7). For this purpose, by means of standard applications of Itô's formula (see, e.g. [46; Theorem 4.4] or [60; Chapter IV, Theorem 3.3]) to the processes  $e^{-rt}(S_t - KX_t)$  and  $e^{-rt}(Q_t - LX_t)$ , we obtain the representations:

$$e^{-rt}(S_t - KX_t) = s - Kx + \int_0^t e^{-ru} (K\delta(S_u, Q_u)X_u - rS_u) du + \int_0^t e^{-ru} dS_u + N_t^1 \quad (2.8)$$

and

$$e^{-rt}(Q_t - LX_t) = q - Lx + \int_0^t e^{-ru} (L\delta(S_u, Q_u)X_u - rQ_u) du + \int_0^t e^{-ru} dQ_u + N_t^2 \quad (2.9)$$

for all  $t \geq 0$ . Here, the processes  $N^i = (N_t^i)_{t \geq 0}$ , for  $i = 1, 2$ , defined by:

$$N_t^1 = -K \int_0^t e^{-ru} \sigma(S_u, Q_u) X_u dB_u \quad \text{and} \quad N_t^2 = -L \int_0^t e^{-ru} \sigma(S_u, Q_u) X_u dB_u \quad (2.10)$$

are continuous uniformly integrable martingales under the probability measure  $P_{x,s,q}$ . Then, inserting  $\tau \wedge \zeta$  in place of  $t$  and applying Doob's optional sampling theorem (see, e.g. [46; Chapter III, Theorem 3.6] and [60; Chapter II, Theorem 3.2]) to the expressions in (2.8) and (2.9), we get that the equalities:

$$\begin{aligned} & E_{x,s,q} [e^{-r\tau}(S_\tau - KX_\tau)I(\tau < \zeta) + e^{-r\zeta}(Q_\zeta - LX_\zeta)I(\zeta \leq \tau)] \\ &= E_{x,s,q} [e^{-r(\tau \wedge \zeta)}(S_{\tau \wedge \zeta} - KX_{\tau \wedge \zeta}) - e^{-r\zeta}(S_\zeta - KX_\zeta - Q_\zeta + LX_\zeta)I(\zeta \leq \tau)] \\ &= s - Kx - E_{x,s,q} [e^{-r\zeta}(S_\zeta - KX_\zeta - Q_\zeta + LX_\zeta)I(\zeta \leq \tau)] \\ &\quad + E_{x,s,q} \left[ \int_0^{\tau \wedge \zeta} e^{-ru} (K\delta(S_u, Q_u)X_u - rS_u) du + \int_0^{\tau \wedge \zeta} e^{-ru} dS_u \right] \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} & E_{x,s,q} [e^{-r\tau}(S_\tau - KX_\tau)I(\tau < \zeta) + e^{-r\zeta}(Q_\zeta - LX_\zeta)I(\zeta \leq \tau)] \\ &= E_{x,s,q} [e^{-r(\tau \wedge \zeta)}(Q_{\tau \wedge \zeta} - LX_{\tau \wedge \zeta}) + e^{-r\tau}(S_\tau - KX_\tau - Q_\tau + LX_\tau)I(\tau < \zeta)] \\ &= q - Lx + E_{x,s,q} [e^{-r\tau}(S_\tau - KX_\tau - Q_\tau + LX_\tau)I(\tau < \zeta)] \\ &\quad + E_{x,s,q} \left[ \int_0^{\tau \wedge \zeta} e^{-ru} (L\delta(S_u, Q_u)X_u - rQ_u) du + \int_0^{\tau \wedge \zeta} e^{-ru} dQ_u \right] \end{aligned} \quad (2.12)$$

hold, for any stopping times  $\tau$  and  $\zeta$ . Hence, it follows from the expressions in (2.11) and (2.12) and the structure of the optimal stopping times in (2.4) and (2.5) that the value function of the optimal stopping game in (2.3) admits the representations:

$$\begin{aligned} V_*(x, s, q) &= s - Kx - E_{x,s,q} [e^{-r\zeta_*}(S_{\zeta_*} - KX_{\zeta_*} - Q_{\zeta_*} + LX_{\zeta_*})I(\zeta_* \leq \tau_*)] \\ &\quad + E_{x,s,q} \left[ \int_0^{\tau_* \wedge \zeta_*} e^{-ru} (K\delta(S_u, Q_u)X_u - rS_u) du + \int_0^{\tau_* \wedge \zeta_*} e^{-ru} dS_u \right] \end{aligned} \quad (2.13)$$

and

$$V_*(x, s, q) = q - Lx + E_{x,s,q} \left[ e^{-r\tau_*} (S_{\tau_*} - KX_{\tau_*} - Q_{\tau_*} + LX_{\tau_*}) I(\tau_* < \zeta_*) \right] \quad (2.14)$$

$$+ E_{x,s,q} \left[ \int_0^{\tau_* \wedge \zeta_*} e^{-ru} (L\delta(S_u, Q_u) X_u - rQ_u) du + \int_0^{\tau_* \wedge \zeta_*} e^{-ru} dQ_u \right]$$

for all  $0 < q \leq x \leq s$ . Here, we denote by  $\tau_* = \tau_*(x, s, q)$  and  $\zeta_* = \zeta_*(x, s, q)$  the optimal stopping times forming a Nash equilibrium in (2.3) for the starting point  $(x, s, q)$  of the process  $(X, S, Q)$ .

Thus, on the one hand, it follows from the structure of the integrand in the first integral of (2.13) and the fact that the second integral there increases whenever the process  $(X, S, Q)$  is located at the diagonal  $d_1 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q \leq x = s\}$  that it should not be optimal for the option holder (maximiser of the expected reward) to exercise the contract when the inequalities  $a'(S_t, Q_t) \vee rS_t/(K\delta(S_t, Q_t)) < X_t \leq S_t$  hold, for any  $t \geq 0$ . Moreover, it follows from the structure of the integrand in first integral of (2.14) and the fact that the second integral there decreases whenever the process  $(X, S, Q)$  is located at the diagonal  $d_2 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q = x \leq s\}$  that it should not be optimal for the option writer (minimiser of the expected reward) to recall the contract when the inequalities  $a'(S_t, Q_t) \vee Q_t \leq X_t < rQ_t/(L\delta(S_t, Q_t))$  hold, for any  $t \geq 0$ . Since the both players are acting simultaneously, these facts mean that the set:

$$C' = \{(x, s, q) \in E' \mid q \vee (rs/(K\delta(s, q))) < x < (rq/(L\delta(s, q))) \wedge s\} \quad (2.15)$$

which may be nonempty due to the assumption that the inequalities  $0 < L < K \leq 1$  hold, represents a part of the continuation region  $C_*$  in (2.6), where we denote by:

$$E' = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q \leq x \leq s \text{ and } x \geq a'(s, q) = (s - q)/(K - L)\} \quad (2.16)$$

the set of all points from the state space  $E$  of the process  $(X, S, Q)$  for which the solution of the original problem of (2.3) may be nontrivial.

On the other hand, it follows from the definition of the process  $(X, S, Q)$  in (1.2)-(1.3) and (1.4) and the form of the first integrand in the reward functional of (2.13), which is equivalent to the one in (2.3), that, for each  $0 < q < s$  fixed, there exists some sufficiently small  $0 < a'(s, q) \leq x \leq rs/(K\delta(s, q)) \wedge s$  such that the point  $(x, s, q)$  belongs to  $D_*$ . Moreover, it follows from the definition of  $(X, S, Q)$  in (1.2)-(1.3) and (1.4) and the form of the first integrand in the reward functional in (2.14), which is equivalent to the one in (2.3), that, for each  $0 < q < s$  fixed, there exists some sufficiently large  $x \geq a'(s, q) \vee q \vee rq/(L\delta(s, q))$  such that the point  $(x, s, q)$  belongs to  $D_*$  too. By means of arguments similar to the ones applied in [13; Subsection 3.2] and [50; Proposition 2.1], these properties can be explained by the facts that the costs for the holder of waiting until the process  $X$  from (1.2) coming from such a small  $x > 0$  increases to the current value of the running maximum process  $S$  and the costs for the writer of waiting until the process  $X$  coming from such a large  $x > 0$  decreases to the current value of the running minimum process  $Q$  may be too large due to the structure of the integrands in the reward functionals of (2.13) and (2.14), respectively.

(iii) Let us finally prove the connectivity of the continuation and stopping regions  $C_*$  and  $D_*$  in (2.6)-(2.7). For this purpose, we recall that the process  $X$  admits the explicit expression

of (1.2) as well as provides a (pathwise) unique strong solution of the stochastic differential equation in (1.3), so that the solutions starting from the different points  $x > 0$  do not intersect each other over the whole infinite time interval. Moreover, we observe from the expression of (1.2) that the discounted asset price process  $(e^{-rt}X_t)_{t \geq 0}$  is a positive supermartingale closed at zero. In this case, by using the facts that the running maximum  $S$  of the process  $X$  from (1.4) started at some point  $x' > 0$  remains less or equal to the running maximum of  $X$  started at  $x \geq x'$ , while the running minimum  $Q$  of  $X$  from (1.4) started at some point  $x'' > 0$  remains greater or equal to the running minimum of  $X$  started at  $x \leq x''$ , taking into account the assumption that the function  $\delta(s, q)$  is increasing in  $s$  and  $q$  on  $[0, \infty]^2$ , we obtain that the inequalities:

$$\begin{aligned} E_{x', s, q} [e^{-r\tau} (S_\tau - K X_\tau) I(\tau < \zeta) + e^{-r\zeta} (Q_\zeta - L X_\zeta) I(\zeta \leq \tau)] - (s - K x') \\ \leq E_{x, s, q} [e^{-r\tau} (S_\tau - K X_\tau) I(\tau < \zeta) + e^{-r\zeta} (Q_\zeta - L X_\zeta) I(\zeta \leq \tau)] - (s - K x) \end{aligned} \quad (2.17)$$

and

$$\begin{aligned} E_{x, s, q} [e^{-r\tau} (S_\tau - K X_\tau) I(\tau < \zeta) + e^{-r\zeta} (Q_\zeta - L X_\zeta) I(\zeta \leq \tau)] - (q - L x) \\ \leq E_{x'', s, q} [e^{-r\tau} (S_\tau - K X_\tau) I(\tau < \zeta) + e^{-r\zeta} (Q_\zeta - L X_\zeta) I(\zeta \leq \tau)] - (q - L x'') \end{aligned} \quad (2.18)$$

hold, for any stopping times  $\tau$  and  $\zeta$  as well as all  $0 < a'(s, q) \vee q \leq x' < x < s$  and  $0 < a'(s, q) \vee q < x < x'' \leq s$ , for  $0 < L < K \leq 1$  given and fixed. Then, when we take some  $(x, s, q) \in C_*$  such that either  $a'(s, q) < x < rs/(K\delta(s, q)) \wedge s$  or  $x > a'(s, q) \vee q \vee rq/(L\delta(s, q))$  holds, it follows from the inequalities in (2.17) and (2.18) as well as from the structure of the value function in (2.3) that the inequalities:

$$V_*(x'', s, q) - (s - K x'') \geq V_*(x, s, q) - (s - K x) > 0 \quad (2.19)$$

and

$$V_*(x', s, q) - (q - L x') \leq V_*(x, s, q) - (q - L x) < 0 \quad (2.20)$$

hold, for all  $0 < a'(s, q) \vee q < x < x'' \leq s$  and  $0 < a'(s, q) \vee q \leq x' < x < s$ , respectively. Hence, when we take some  $(x, s, q) \in D_*$  such that either  $a'(s, q) < x < rs/(K\delta(s, q)) \wedge s$  or  $x > a'(s, q) \vee q \vee rq/(L\delta(s, q))$  holds, it follows from the inequalities in (2.17) and (2.18) as well as from the structure of the value function in (2.3) that the inequalities:

$$V_*(x', s, q) - (s - K x') \leq V_*(x, s, q) - (s - K x) = 0 \quad (2.21)$$

and

$$V_*(x'', s, q) - (q - L x'') \geq V_*(x, s, q) - (q - L x) = 0 \quad (2.22)$$

hold, for all  $0 < a'(s, q) \vee q \leq x' < x < s$  and  $0 < a'(s, q) \vee q < x < x'' \leq s$ , respectively. Thus, combining these arguments together with the facts deduced in Parts (i)-(ii) above and checking with the comments in [13; Subsection 3.3] and [50; Subsection 3.3], we may therefore conclude that there exist functions  $a_*(s, q)$  and  $b_*(s, q)$  satisfying the inequalities  $a'(s, q) \leq$



$a_*(s, q) \leq rs/(K\delta(s, q)) \wedge s$  and  $b_*(s, q) \geq a'(s, q) \vee q \vee rq/(K\delta(s, q))$ , for  $0 < q < s$ , such that the continuation region  $C_*$  in (2.6) has the form:

$$C_* = \{(x, s, q) \in E' \mid a_*(s, q) < x < b_*(s, q)\} \quad (2.23)$$

while the stopping region  $D_*$  in (2.7) is given by:

$$D_* = \{(x, s, q) \in E \mid \text{either } x \leq a_*(s, q) \text{ or } x \geq b_*(s, q)\}. \quad (2.24)$$

(iv) In order to determine some upper and lower bounds for the value function in (2.3) and optimal stopping boundaries in (2.23) and (2.24), we consider the optimal stopping problems with the value functions  $\bar{V}(x, s, q)$  and  $\underline{V}(x, s, q)$  from (5.1). It is shown in Section 5 below that the functions  $\bar{V}(x, s, q)$  and  $\underline{V}(x, s, q)$  admit the explicit expressions in (5.11) and (5.12), while the associated optimal stopping times  $\bar{\tau}$  and  $\underline{\zeta}$  are given by (5.2), where the boundaries  $\underline{a}(s, q)$  and  $\bar{b}(s, q)$  are determined as the maximal and minimal solutions of the first-order nonlinear ordinary differential equations in (3.23) and (3.26) staying below or above the diagonals  $d_1 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q \leq x = s\}$  and  $d_2 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q = x \leq s\}$ , respectively. If we suppose that either the inequality  $a_*(s, q) < \underline{a}(s, q)$  or  $b_*(s, q) > \bar{b}(s, q)$  holds, then, for each  $x > 0$  such that either  $x \in (a_*(s, q), \underline{a}(s, q))$  or  $x \in (\bar{b}(s, q), b_*(s, q))$  given and fixed, we would have either  $V_*(x, s, q) > s - Kx = \bar{V}(x, s, q)$  or  $V_*(x, s, q) < q - Lx = \underline{V}(x, s, q)$ , respectively, contradicting the obvious facts that the inequalities  $V_*(x, s, q) \leq \bar{V}(x, s, q)$  and  $V_*(x, s, q) \geq \underline{V}(x, s, q)$  hold, for all  $(x, s, q) \in E'$ . Thus, we may conclude that the inequalities  $a_*(s, q) \geq \underline{a}(s, q)$  and  $b_*(s, q) \leq \bar{b}(s, q)$  should be satisfied, for all  $0 < q < s$  (see Figures 1 and 2 below for computer drawings of the optimal stopping boundaries  $a_*(s, q)$  and  $b_*(s, q)$  as well as their estimates  $\underline{a}(s, q)$  and  $\bar{b}(s, q)$ ).

**2.3 The three-dimensional free-boundary problem.** By means of standard arguments based on an application of Itô's formula, it is shown that the infinitesimal operator  $\mathbb{L}$  of the process  $(X, S, Q)$  has the form:

$$\mathbb{L} = (r - \delta(s, q)) x \partial_x + \frac{\sigma^2(s, q)x^2}{2} \partial_{xx} \quad \text{in } 0 < q < x < s \quad (2.25)$$

$$\partial_s = 0 \quad \text{at } 0 < q \leq x = s \quad \text{and} \quad \partial_q = 0 \quad \text{at } 0 < q = x \leq s \quad (2.26)$$

(see, e.g. [50; Subsection 3.1]). In order to find analytic expressions for the unknown value functions  $V_*(x, s, q)$  from (2.3) and the unknown boundaries  $a_*(s, q)$  and  $b_*(s, q)$  from (2.23)-(2.24), let us build on the results of general theory of optimal stopping problems for Markov processes (see, e.g. [57; Chapter IV, Section 8]). We can reduce the optimal stopping game of (2.3) to the equivalent coupled free-boundary problem for  $V(x, s, q)$  with  $a(s, q)$  and  $b(s, q)$  given by:

$$(\mathbb{L}V - rV)(x, s, q) = 0 \quad \text{for } q \vee a(s, q) < x < b(s, q) \wedge s \quad (2.27)$$

$$V(x, s, q)|_{x=a(s, q)+} = s - K a(s, q), \quad V(x, s, q)|_{x=b(s, q)-} = q - L b(s, q) \quad (2.28)$$

$$V(x, s, q) = s - Kx \quad \text{for } x < a(s, q), \quad V(x, s, q) = q - Lx \quad \text{for } x > b(s, q) \quad (2.29)$$

$$s - Kx < V(x, s, q) < q - Lx \quad \text{for } a(s, q) < x < b(s, q) \quad (2.30)$$

$$(\mathbb{L}V - rV)(x, s, q) < 0 \quad \text{for } x < a(s, q), \quad (\mathbb{L}V - rV)(x, s, q) > 0 \quad \text{for } x > b(s, q) \quad (2.31)$$



where the *instantaneous-stopping* conditions in (2.28) are satisfied, when  $a(s, q) \geq q$  or  $b(s, q) \leq s$  holds, for each  $0 < q < s$ . Moreover, we further assume that the *smooth-fit* conditions:

$$\partial_x V(x, s, q)|_{x=a(s, q)+} = -K, \quad \partial_x V(x, s, q)|_{x=b(s, q)-} = -L \quad (2.32)$$

are satisfied, when  $a(s, q) \geq q$  or  $b(s, q) \leq s$  holds, while the *normal-reflection* conditions:

$$\partial_s V(x, s, q)|_{x=s-} = 0, \quad \partial_q V(x, s, q)|_{x=q+} = 0 \quad (2.33)$$

are satisfied, when  $b(s, q) > s$  or  $a(s, q) < q$  holds, for each  $0 < q < s$ , respectively. On the one hand, when either the inequality  $a(s, q) \geq q$  or  $b(s, q) \leq s$  holds, for some  $0 < q < s$ , the continuous process  $X$  can cross the left-hand boundary  $a(S, Q)$  before hitting the lower diagonal  $d_2$  or cross the right-hand boundary  $b(S, Q)$  before hitting the upper diagonal  $d_1$ , so that we can assume that the left-hand or the right-hand smooth-fit conditions of (2.32) are satisfied for the candidate value function  $V(x, s, q)$  at  $a(s, q)$  or  $b(s, q)$ , respectively. On the other hand, when either the inequality  $b(s, q) > s$  or  $a(s, q) < q$  holds, for some  $0 < q < s$ , the process  $X$  can hit the upper diagonal  $d_1 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q \leq x = s\}$  before crossing the right-hand boundary  $b(S, Q)$  or hit the lower diagonal  $d_2 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q = x \leq s\}$  before crossing the left-hand boundary  $a(S, Q)$ , so that we can assume that the left-hand or the right-hand normal-reflection conditions of (2.33) are satisfied for  $V(x, s, q)$  at  $d_1$  and  $d_2$ , respectively. These properties are verified in the proof of Theorem 4.1 below, while the inequalities in (2.31) follow directly from the arguments of Part (ii) of Subsection 2.2 above.

### 3. Solutions to the free-boundary problem

In this section, we obtain closed-form expressions for the value function  $V_*(x, s, q)$  in (2.3) associated with the perpetual floating-strike lookback game option and derive first-order non-linear ordinary differential equations for the optimal stopping boundaries  $a_*(s, q)$  and  $b_*(s, q)$  from (2.23)-(2.24) forming a solution to the free-boundary problem in (2.27)-(2.31) with (2.32) and (2.33).

**3.1 The candidate value function.** We first observe that the general solution of the second-order ordinary differential equation in (2.27) has the form:

$$V(x, s, q) = C_1(s, q) x^{\gamma_1(s, q)} + C_2(s, q) x^{\gamma_2(s, q)} \quad (3.1)$$

where  $C_i(s, q)$ , for  $i = 1, 2$ , are some arbitrary continuously differentiable functions and  $\gamma_i(s, q)$ , for  $i = 1, 2$ , are given by:

$$\gamma_i(s, q) = \frac{1}{2} - \frac{r - \delta(s, q)}{\sigma^2(s, q)} - (-1)^i \sqrt{\left(\frac{1}{2} - \frac{r - \delta(s, q)}{\sigma^2(s, q)}\right)^2 + \frac{2r}{\sigma^2(s, q)}} \quad (3.2)$$

so that  $\gamma_2(s, q) < 0 < 1 < \gamma_1(s, q)$  holds, for all  $0 < q < s$ . Then, by applying the instantaneous-stopping conditions from (2.28) to the function in (3.1), we get that the equalities:

$$C_1(s, q) a^{\gamma_1(s, q)}(s, q) + C_2(s, q) a^{\gamma_2(s, q)}(s, q) = s - K a(s, q) \quad (3.3)$$

$$C_1(s, q) b^{\gamma_1(s, q)}(s, q) + C_2(s, q) b^{\gamma_2(s, q)}(s, q) = q - L b(s, q) \quad (3.4)$$

are satisfied, when  $a(s, q) \geq q$  and  $b(s, q) \leq s$  holds, for each  $0 < q < s$ , respectively. Hence, by using the smooth-fit conditions from (2.32), we obtain that the equalities:

$$C_1(s, q) \gamma_1(s, q) a^{\gamma_1(s, q)}(s, q) + C_2(s, q) \gamma_2(s, q) a^{\gamma_2(s, q)}(s, q) = -K a(s, q) \quad (3.5)$$

$$C_1(s, q) \gamma_1(s, q) b^{\gamma_1(s, q)}(s, q) + C_2(s, q) \gamma_2(s, q) b^{\gamma_2(s, q)}(s, q) = -L b(s, q) \quad (3.6)$$

are satisfied, when  $a(s, q) \geq q$  and  $b(s, q) \leq s$  holds, for each  $0 < q < s$ , respectively. Thus, by applying the normal-reflection conditions from (2.33) to the function in (3.1), we obtain that the equalities:

$$\sum_{i=1}^2 \left( \partial_s C_i(s, q) s^{\gamma_i(s, q)} + C_i(s, q) \partial_s \gamma_i(s, q) s^{\gamma_i(s, q)} \ln s \right) = 0 \quad (3.7)$$

$$\sum_{i=1}^2 \left( \partial_q C_i(s, q) q^{\gamma_i(s, q)} + C_i(s, q) \partial_q \gamma_i(s, q) q^{\gamma_i(s, q)} \ln q \right) = 0 \quad (3.8)$$

are satisfied, when  $b(s, q) > s$  and  $a(s, q) < q$  holds, for each  $0 < q < s$ , respectively. Here, the partial derivatives  $\partial_s \gamma_i(s, q)$  and  $\partial_q \gamma_i(s, q)$  take the form:

$$\partial_s \gamma_i(s, q) = \varphi(s, q) - (-1)^i \frac{\sigma(s, q)(\gamma_1(s, q) + \gamma_2(s, q))\sigma^3(s, q) - 4r\partial_s \sigma(s, q)}{\sigma^2(s, q)\sqrt{(\gamma_1(s, q) + \gamma_2(s, q))^2\sigma^2(s, q) + 8r}} \quad (3.9)$$

$$\partial_q \gamma_i(s, q) = \psi(s, q) - (-1)^i \frac{\psi(s, q)(\gamma_1(s, q) + \gamma_2(s, q))\sigma^3(s, q) - 4r\partial_q \sigma(s, q)}{\sigma^2(s, q)\sqrt{(\gamma_1(s, q) + \gamma_2(s, q))^2\sigma^2(s, q) + 8r}} \quad (3.10)$$

for  $i = 1, 2$ , and the functions  $\varphi(s, q)$  and  $\psi(s, q)$  are defined by:

$$\varphi(s, q) = \frac{\sigma(s, q)\partial_s \delta(s, q) + 2(r - \delta(s, q))\partial_s \sigma(s, q)}{\sigma^3(s, q)} \quad (3.11)$$

$$\psi(s, q) = \frac{\sigma(s, q)\partial_q \delta(s, q) + 2(r - \delta(s, q))\partial_q \sigma(s, q)}{\sigma^3(s, q)} \quad (3.12)$$

for  $0 < q < s$ .

Now, by solving the system of equations in (3.3)+(3.4), we obtain that the function in (3.1) admits the representation:

$$V(x, s, q; a(s, q), b(s, q)) = \sum_{i=1}^2 C_i(s, q; a(s, q), b(s, q)) x^{\gamma_i(s, q)} \quad (3.13)$$

for  $0 < q \leq a(s, q) < x < b(s, q) \leq s$ , where

$$C_i(s, q; a(s, q), b(s, q)) = \frac{(s - Ka(s, q))b^{\gamma_{3-i}(s, q)}(s, q) - (q - Lb(s, q))a^{\gamma_{3-i}(s, q)}(s, q)}{a^{\gamma_i(s, q)}(s, q)b^{\gamma_{3-i}(s, q)}(s, q) - b^{\gamma_i(s, q)}(s, q)a^{\gamma_{3-i}(s, q)}(s, q)} \quad (3.14)$$

when  $0 < q \leq a(s, q) < b(s, q) \leq s$  holds, for every  $i = 1, 2$ . Then, by solving the system of equations in (3.3)+(3.5), we obtain that the function in (3.1) admits the representation:

$$V(x, s, q; a(s, q)) = C_1(s, q; a(s, q)) x^{\gamma_1(s, q)} + C_2(s, q; a(s, q)) x^{\gamma_2(s, q)} \quad (3.15)$$

for  $0 < q \leq a(s, q) < x \leq s < b(s, q)$ , where

$$C_i(s, q; a(s, q)) = \frac{\gamma_{3-i}(s, q)(s - Ka(s, q)) + Ka(s, q)}{(\gamma_{3-i}(s, q) - \gamma_i(s, q))a^{\gamma_i(s, q)}(s, q)} \quad (3.16)$$

when  $0 < q \leq a(s, q) < s < b(s, q)$  holds, for every  $i = 1, 2$ . Also, by solving the system of equations in (3.4)+(3.6), we obtain that the function in (3.1) admits the representation:

$$V(x, s, q; b(s, q)) = C_1(s, q; b(s, q)) x^{\gamma_1(s, q)} + C_2(s, q; b(s, q)) x^{\gamma_2(s, q)} \quad (3.17)$$

for  $0 < a(s, q) < q \leq x < b(s, q) \leq s$ , where

$$C_i(s, q; b(s, q)) = \frac{\gamma_{3-i}(s, q)(q - Lb(s, q)) + Lb(s, q)}{(\gamma_{3-i}(s, q) - \gamma_i(s, q))b^{\gamma_i(s, q)}(s, q)} \quad (3.18)$$

when  $0 < a(s, q) < q < b(s, q) \leq s$  holds, for every  $i = 1, 2$ .

Finally, by means of straightforward computations, it can be deduced from the expression in (3.13) that the first-order and second-order partial derivatives  $\partial_x V(x, s, q; a(s, q), b(s, q))$  and  $\partial_{xx} V(x, s, q; a(s, q), b(s, q))$  of the function  $V(x, s, q; a(s, q), b(s, q))$  take the form:

$$\partial_x V(x, s, q; a(s, q), b(s, q)) = \sum_{i=1}^2 C_i(s, q; a(s, q), b(s, q)) \gamma_i(s, q) x^{\gamma_i(s, q)-1} \quad (3.19)$$

and

$$\partial_{xx} V(x, s, q; a(s, q), b(s, q)) = \sum_{i=1}^2 C_i(s, q; a(s, q), b(s, q)) \gamma_i(s, q)(\gamma_i(s, q) - 1) x^{\gamma_i(s, q)-2} \quad (3.20)$$

for  $0 < q \vee a(s, q) < x < b(s, q) \wedge s$ . Note that the same first-order and second-order partial derivatives of the functions  $V(x, s, q; a(s, q))$ , for  $0 < q \leq a(s, q) < s < b(s, q)$ , and  $V(x, s, q; b(s, q))$ , for  $0 < a(s, q) < q < b(s, q) \leq s$ , from (3.15) with (3.16) and (3.17) with (3.18) are computed similarly.

**3.2 The candidate stopping boundaries.** We now apply the conditions of (3.5)-(3.6) to the functions  $C_i(s, q; a(s, q), b(s, q))$ , for  $i = 1, 2$ , in (3.14) to obtain the equalities:

$$\frac{\gamma_i(s, q)(s - Ka(s, q)) + Ka(s, q)}{\gamma_i(s, q)(q - Lb(s, q)) + Lb(s, q)} = \left( \frac{a(s, q)}{b(s, q)} \right)^{\gamma_{3-i}(s, q)} \quad (3.21)$$

for  $0 < a'(s, q) \vee q \leq a(s, q) < b(s, q) \leq s$ , while otherwise, we set  $a(s, q) = a'(s, q)$  and apply the condition of (3.6) to the same functions to obtain the equality:

$$\begin{aligned} & \sum_{i=1}^2 \frac{(s - Ka'(s, q))b^{\gamma_{3-i}(s, q)}(s, q) - (q - Lb(s, q))a'^{\gamma_{3-i}(s, q)}(s, q)}{a'^{\gamma_i(s, q)}(s, q)b^{\gamma_{3-i}(s, q)}(s, q) - b^{\gamma_i(s, q)}(s, q)a'^{\gamma_{3-i}(s, q)}(s, q)} \gamma_i(s, q) b^{\gamma_i(s, q)}(s, q) \\ & = -Lb(s, q) \end{aligned} \quad (3.22)$$

for  $0 < a(s, q) < a'(s, q) \vee q < b(s, q) \leq s$  and  $i = 1, 2$ . The existence and uniqueness of solutions of the system of arithmetic equations in (3.21) as well as of the equation in (3.22) on the admissible intervals follow from the arguments of Subsection 3.4 below. Observe that the system of arithmetic equations in (3.21) as well as the equation in (3.22) satisfy the conditions of the classical (two-dimensional) implicit function theorem, so that the resulting solutions  $a_*(s, q)$  and  $b_*(s, q)$  turn out to be continuously differentiable. Furthermore, assuming that the candidate boundary functions  $a(s, q)$  and  $b(s, q)$  are continuously differentiable, we apply the condition of (3.7) to the functions  $C_i(s, q; a(s, q))$ , for  $i = 1, 2$ , in (3.16) to conclude that the candidate boundary  $a(s, q)$  satisfies the ordinary differential equation:

$$\partial_s a(s, q) = \sum_{j=1}^2 \frac{C_i(s, q; a(s, q)) \partial_s \gamma_i(s, q) s^{\gamma_i(s, q)} \ln s + \Psi_{1,i}(a(s, q), s, q) (s/a(s, q))^{\gamma_i(s, q)}}{\Phi_1(a(s, q), s, q) ((s/a(s, q))^{\gamma_1(s, q)} - (s/a(s, q))^{\gamma_2(s, q)})} \quad (3.23)$$

for  $0 < a'(s, q) \vee q \leq a(s, q) < s < b(s, q)$ , where we set:

$$\Phi_1(x, s, q) = \frac{(\gamma_1(s, q) + \gamma_2(s, q))K + \gamma_1(s, q)\gamma_2(s, q)(s - Kx)/x}{\gamma_2(s, q) - \gamma_1(s, q)} \quad (3.24)$$

and

$$\begin{aligned} \Psi_{1,i}(x, s, q) &= \frac{\partial_s \gamma_{3-i}(s, q)(s - Kx) + \gamma_{3-i}(s, q)}{\gamma_{3-i}(s, q) - \gamma_i(s, q)} \\ &\quad - \frac{(Kx + \gamma_{3-i}(s, q)(s - Kx))(\partial_s \gamma_i(s, q) \ln x + \partial_s \ln(\gamma_{3-i}(s, q) - \gamma_i(s, q)))}{\gamma_{3-i}(s, q) - \gamma_i(s, q)} \end{aligned} \quad (3.25)$$

for all  $0 < q \leq x \leq s$  and every  $i = 1, 2$ . We also apply the condition of (3.8) to the functions  $C_i(s, q; b(s, q))$ , for  $i = 1, 2$ , in (3.18) to conclude that the candidate boundary  $b(s, q)$  satisfies the ordinary differential equation:

$$\partial_q b(s, q) = \sum_{i=1}^2 \frac{C_i(s, q; b(s, q)) \partial_q \gamma_i(s, q) q^{\gamma_i(s, q)} \ln q + \Psi_{2,i}(b(s, q), s, q) (q/b(s, q))^{\gamma_i(s, q)}}{\Phi_2(b(s, q), s, q) ((q/b(s, q))^{\gamma_1(s, q)} - (q/b(s, q))^{\gamma_2(s, q)})} \quad (3.26)$$

for  $0 < a'(s, q) \leq a(s, q) < q < b(s, q) \leq s$ , where we set:

$$\Phi_2(x, s, q) = \frac{(\gamma_1(s, q) + \gamma_2(s, q))L + \gamma_1(s, q)\gamma_2(s, q)(q - Lx)/x}{\gamma_2(s, q) - \gamma_1(s, q)} \quad (3.27)$$

and

$$\begin{aligned} \Psi_{2,i}(x, s, q) &= \frac{\partial_q \gamma_{3-i}(s, q)(q - Lx) + \gamma_{3-i}(s, q)}{\gamma_{3-i}(s, q) - \gamma_i(s, q)} \\ &\quad - \frac{(Lx + \gamma_{3-i}(s, q)(q - Lx))(\partial_q \gamma_i(s, q) \ln x + \partial_q \ln(\gamma_{3-i}(s, q) - \gamma_i(s, q)))}{\gamma_{3-i}(s, q) - \gamma_i(s, q)} \end{aligned} \quad (3.28)$$

for all  $0 < q \leq x \leq s$  and every  $i = 1, 2$ . Note that, by virtue of the assumptions on the coefficients  $\delta(s, q) > 0$  and  $\sigma(s, q) > 0$  of the diffusion-type process  $X$  from (1.2)-(1.3) and (1.4), the right-hand sides of the expressions in (3.23) with (3.24)-(3.25) and (3.26) with

(3.27)-(3.28) are (locally) continuous in  $(s, q, a(s, q))$  and  $(s, q, b(s, q))$  and (locally) Lipschitz in  $a(s, q)$  and  $b(s, q)$ , for each  $0 < q < s$  fixed. Thus, by means of the classical results on the existence and uniqueness of solutions for first-order nonlinear ordinary differential equations, the equations in (3.23) and (3.26) admit (locally) unique solutions, which can be constructed by means of Picard's method of successive approximations (see Subsection 5.3 below for further constructions and references).

**3.3 The structure of continuation region.** In order to specify the optimal exercise boundaries for the floating-strike lookback game options, let us consider the functions  $a_*(s, q)$  and  $b_*(s, q)$ , which either provide a unique solution to the system of arithmetic equations in (3.21) such that  $a'(s, q) \leq a_*(s, q) < b_*(s, q)$ , or the function  $b_*(s, q) > a_*(s, q) = a'(s, q)$  represents the largest root of the equation in (3.6), or otherwise, the property  $a_*(s, q) = b_*(s, q) = a'(s, q)$  holds, for each  $0 < q < s$  fixed. On the one hand, we can set  $s_0^*(q) = \infty$  and define the functions  $s_{2k-1}^*(q) = \sup\{s < s_{2k-2}^*(q) \mid b_*(s, q) > s\}$  and  $s_{2k}^*(q) = \sup\{s < s_{2k-1}^*(q) \mid b_*(s, q) < s\}$ , whenever they exist, and put  $s_{2k-1}^*(q) = s_{2k}^*(q) = 0$  otherwise, so that the inequalities  $0 \leq s_{2k-1}^*(q) \leq s_{2k-2}^*(q) \leq \infty$  hold, for all  $k \in \mathbb{N}$ , and each  $q > 0$  fixed. In other words, the boundary  $b_*(s, q)$  exits the region  $E$  from the side of the diagonal  $d_1 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q \leq x = s\}$  passing through the points  $(s_{2k-1}^*(q), s_{2k-1}^*(q), q)$  and comes back to  $E$  from the side of  $d_1$  passing through the points  $(s_{2k}^*(q), s_{2k}^*(q), q)$ , for  $k \in \mathbb{N}$ , for each  $q > 0$  fixed. Hence, the candidate value function  $V(x, s, q; a_*(s, q), b_*(s, q))$  admits the representation of (3.13) with (3.14) and the candidate stopping boundaries  $a_*(s, q)$  and  $b_*(s, q)$  solve the system of arithmetic equations in (3.21) in the regions:

$$\tilde{R}_{2k-1}(a_*, b_*) = \{(x, s, q) \in E' \mid s_{2k-1}^*(q) \leq s < s_{2k-2}^*(q)\} \quad (3.29)$$

while the candidate value function  $V(x, s, q; a_*(s, q))$  admits the representation of (3.15) with (3.16) and the candidate stopping boundary  $a_*(s, q)$  either solves the first-order nonlinear ordinary differential equation in the regions:

$$\tilde{R}_{2k}(a_*) = \{(x, s, q) \in E' \mid s_{2k}^*(q) \leq s < s_{2k-1}^*(q)\} \quad (3.30)$$

both representing subsets of the continuation region  $C_*$  in (2.23), or coincides with  $a'(s, q)$ , for each  $q > 0$  fixed, and every  $k \in \mathbb{N}$ . Furthermore, we observe that, the process  $(X, S, Q)$  can enter the region  $\tilde{R}_{2k-1}(a_*, b_*)$  in (3.30) from the region  $\tilde{R}_{2k}(a_*)$  in (3.29) only through the point  $(s_{2k-1}^*(q), s_{2k-1}^*(q), q)$ , for any  $k \in \mathbb{N}$ , by hitting the plane  $d_1 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q \leq x = s\}$ , so that by increasing its second component  $S$ . In this respect, the candidate value function should satisfy the instantaneous-stopping and smooth-fit conditions at the points  $(s_{2k-1}^*(q), s_{2k-1}^*(q), q)$ , that is expressed by the equalities:

$$\sum_{i=1}^2 C_i(s_{2k-1}^*(q)-, q; a_*(s_{2k-1}^*(q)-, q)) (s_{2k-1}^*(q))^{\gamma_i(s_{2k-1}^*(q), q)} \quad (3.31)$$

$$= V(s_{2k-1}^*(q), s_{2k-1}^*(q), q; a_*(s_{2k-1}^*(q), q), b_*(s_{2k-1}^*(q), q))$$

$$\sum_{i=1}^2 C_i(s_{2k-1}^*(q)-, q; a_*(s_{2k-1}^*(q)-, q)) \gamma_i(s_{2k-1}^*(q)-, q) (s_{2k-1}^*(q))^{\gamma_i(s_{2k-1}^*(q), q)} \quad (3.32)$$

$$= s_{2k-1}^*(q) \partial_x V(s_{2k-1}^*(q), s_{2k-1}^*(q), q; a_*(s_{2k-1}^*(q), q), b_*(s_{2k-1}^*(q), q))$$

where the functions  $C_i(s, q; a_*(s, q))$ , for  $i = 1, 2$ , are given by (3.16) and the function  $V(x, s, q; a_*(s, q), b_*(s, q))$  has the form of (3.13) with (3.14), while the boundary  $a_*(s, q)$  provides a unique solution of the first-order nonlinear ordinary differential equation in (3.23) in the region  $\hat{R}_{2k}(a_*)$  from (3.30) satisfying the (starting) conditions of (3.31)-(3.32) above, for each  $q > 0$  fixed, and every  $k \in \mathbb{N}$ , respectively.

On the other hand, we can set  $q_0^*(s) = 0$  and define the functions  $q_{2l-1}^*(s) = \inf\{q > q_{2l-2}^*(s) \mid a_*(s, q) < q\}$  and  $q_{2l}^*(s) = \inf\{q > q_{2l-1}^*(s) \mid a_*(s, q) > q\}$ , whenever they exist, and put  $q_{2l-1}^*(s) = q_{2l}^*(s) = \infty$  otherwise, so that the inequalities  $0 \leq q_{2l-2}^*(s) \leq q_{2l-1}^*(s) \leq \infty$  hold, for all  $l \in \mathbb{N}$ , and each  $s > 0$  fixed. In other words, the boundary  $a_*(s, q)$  exits the region  $E$  from the side of the diagonal  $d_2 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q = x \leq s\}$  passing through the points  $(q_{2l-1}^*(s), s, q_{2l-1}^*(q))$  and comes back to  $E$  from the side of the diagonal  $d_2$  passing through the points  $(q_{2l-1}^*(s), s, q_{2l-1}^*(s))$ , for  $k \in \mathbb{N}$ , for each  $s > 0$  fixed. Hence, the candidate value function  $V(x, s, q; a_*(s, q), b_*(s, q))$  admits the representation of (3.13) with (3.14) and the candidate stopping boundaries  $a_*(s, q)$  and  $b_*(s, q)$  solve the system of arithmetic equations in (3.21) in the regions:

$$\hat{R}_{2l-1}(a_*, b_*) = \{(x, s, q) \in E' \mid q_{2l-2}^*(s) < q \leq q_{2l-1}^*(s)\} \quad (3.33)$$

while the candidate value function  $V(x, s, q; b_*(s, q))$  admits the representation of (3.17) with (3.18) and the candidate stopping boundary  $b_*(s, q)$  solves the first-order nonlinear ordinary differential equation in the regions:

$$\hat{R}_{2l}(b_*) = \{(x, s, q) \in E' \mid q_{2l-1}^*(s) < q \leq q_{2l}^*(s)\} \quad (3.34)$$

both representing subsets of the continuation region  $C_*$  in (2.23), for each  $s > 0$  fixed, and every  $l \in \mathbb{N}$ . Furthermore, we observe that, the process  $(X, S, Q)$  can enter the region  $\hat{R}_{2l-1}(a_*, b_*)$  in (3.34) from the region  $\hat{R}_{2l}(b_*)$  in (3.33) only through the point  $(q_{2l-1}^*(s), s, q_{2l-1}^*(s))$ , for any  $l \in \mathbb{N}$ , by hitting the plane  $d_2 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q = x \leq s\}$ , so that by decreasing its third component  $Q$ . In this respect, the candidate value function should satisfy the instantaneous-stopping and smooth-fit conditions at the points  $(q_{2l-1}^*(s), s, q_{2l-1}^*(s))$ , that is expressed by the equalities:

$$\sum_{i=1}^2 C_i(s, q_{2l-1}^*(s)+; b_*(s, q_{2l-1}^*(s)+)) (q_{2l-1}^*(s))^{\gamma_j(s, q_{2l-1}^*(s))} \quad (3.35)$$

$$\begin{aligned} &= V(q_{2l-1}^*(s), s, q_{2l-1}^*(s); a_*(s, q_{2l-1}^*(s)), b_*(s, q_{2l-1}^*(s))) \\ &\sum_{i=1}^2 C_i(s, q_{2l-1}^*(s)+; b_*(s, q_{2l-1}^*(s)+)) \gamma_i(s, q_{2l-1}^*(s)) (q_{2l-1}^*(s))^{\gamma_i(s, q_{2l-1}^*(s))} \\ &= q_{2l-1}^*(s) \partial_x V(q_{2l-1}^*(s), s, q_{2l-1}^*(s); a_*(s, q_{2l-1}^*(s)), b_*(s, q_{2l-1}^*(s))) \end{aligned} \quad (3.36)$$

where the functions  $C_i(s, q; b_*(s, q))$ , for  $i = 1, 2$ , are given by (3.18) and the function  $V(x, s, q; a_*(s, q), b_*(s, q))$  has the form of (3.13) with (3.14), while the boundary  $b_*(s, q)$  provides a unique solution of the first-order nonlinear ordinary differential equation in (3.26) in the region  $\hat{R}_{2l}(b_*)$  from (3.34) satisfying the (starting) conditions of (3.35)-(3.36) above, for each  $s > 0$  fixed, and every  $k \in \mathbb{N}$ , respectively. Note that the process  $(X, S, Q)$  cannot come



from the region  $\tilde{R}_{2k}(a_*)$  in (3.30) directly to the region  $\hat{R}_{2l}(b_*)$  in (3.34) and vice versa without crossing the regions  $\hat{R}_{2k}(a_*, b_*)$  in (3.29) or  $\hat{R}_{2l}(a_*, b_*)$  in (3.33), for every  $k, l \in \mathbb{N}$ , respectively.

Finally, we observe that if we have  $\gamma_i(s, q) = \gamma_i(q)$ , for  $i = 1, 2$ , in (3.2), then the appropriate left-hand exercise boundary for the floating-strike lookback game option in (2.23)-(2.24) takes the form  $a_*(s, q) = a'(s, q) \vee g_*(q)s$  with some function  $0 < g_*(q) < 1$ , while if we have  $\gamma_i(s, q) = \gamma_i(s)$ , for  $i = 1, 2$ , in (3.2), then the appropriate right-hand exercise boundary for that contract there takes the form  $b_*(s, q) = a'(s, q) \vee h_*(s)q$  with some function  $h_*(s) > 1$ , for all  $0 < q < s$ . In these cases, we have the sole regions  $\tilde{R}_1(a_*, b_*)$ ,  $\tilde{R}_2(a_*)$  in (3.29)-(3.30) with  $k = 1$  and  $\hat{R}_1(a_*, b_*)$ ,  $\hat{R}_2(b_*)$  in (3.33)-(3.34) with  $l = 1$ , respectively. Moreover, if we have  $\gamma_i(s, q) = \gamma_i$ , for  $i = 1, 2$ , in (3.2), then we have  $a_*(s, q) = a'(s, q) \vee a_*(s) = a'(s, q) \vee g_*s$  and  $b_*(s, q) = a'(s, q) \vee b_*(q) = a'(s, q) \vee h_*q$  with some constants  $0 < g_* < 1$  and  $h_* > 1$ , for all  $0 < q < s$ . The latter property can be explained by the fact that the original problem of (1.1), and thus, the three-dimensional problem of (2.3) can be reduced to an optimal stopping zero-sum game for the process  $(S/X, Q/X) = (S_t/X_t, Q_t/X_t)_{t \geq 0}$  representing a two-dimensional Markov diffusion process with reflection, by means of the change-of-measure arguments from [63] (see also [21]).

**3.4 The system of arithmetic equations.** We finally extend the arguments from [24; Example 4.2] (see also [18; Section 3] and [58; Theorem 1]) to show that the system of arithmetic power equations in (3.21) admits a unique solution. For this purpose, by virtue of straightforward calculations, we first observe that the system of equations in (3.21) is equivalent to the one:

$$\Xi_i(a(s, q); s, q) = \Upsilon_i(b(s, q); s, q) \quad (3.37)$$

for  $0 < a'(s, q) \vee q \leq a(s, q) < b(s, q) \leq s$ , where we set:

$$\Xi_i(a; s, q) = \frac{\gamma_i(s, q)(s - Ka) + Ka}{a^{\gamma_{3-i}(s, q)}} \quad \text{and} \quad \Upsilon_i(b; s, q) = \frac{\gamma_i(s, q)(q - Lb) + Lb}{b^{\gamma_{3-i}(s, q)}} \quad (3.38)$$

for all  $0 < q < a < b < s$  and every  $i = 1, 2$ .

In order to show the existence and uniqueness of a solution of the system of arithmetic power equations in (3.37) with (3.38), we develop the idea of proof of the existence and uniqueness of solutions applied to the systems of arithmetic power equations in [24; Example 4.2] (see also the systems (4.73)-(4.74) in [65; Chapter IV, Section 2], the system (3.16)-(3.17) in [18; Section 3], and [58; Theorem 1]). For this purpose, we observe that, for the derivatives of the functions  $\Xi_i(a)$  and  $\Upsilon_i(b)$ , for  $i = 1, 2$ , defined in (3.37), the expressions:

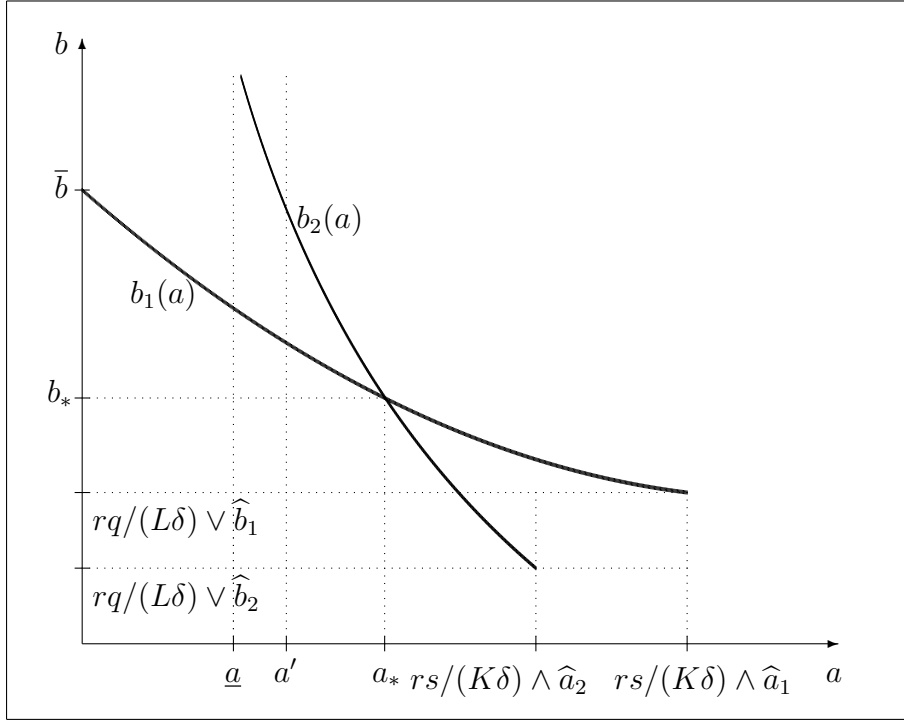
$$\Xi'_i(a; s, q) = \frac{(\gamma_1(s, q) - 1)(\gamma_2(s, q) - 1)K(a - rs/(K\delta(s, q)))}{a^{\gamma_{3-i}(s, q)+1}} \quad (3.39)$$

and

$$\Upsilon'_i(b; s, q) = \frac{(\gamma_1(s, q) - 1)(\gamma_2(s, q) - 1)L(b - rq/(L\delta(s, q)))}{b^{\gamma_{3-i}(s, q)+1}} \quad (3.40)$$

hold, so that the following inequalities:

$$\Xi'_i(a; s, q) > 0 \quad \text{for} \quad a < \frac{rs}{K\delta(s, q)} \quad \text{and} \quad \Upsilon'_i(b; s, q) < 0 \quad \text{for} \quad b > \frac{rq}{L\delta(s, q)} \quad (3.41)$$



**Figure 3.** A computer drawing of the boundary functions  $b_1(a)$  and  $b_2(a)$  in the case  $a'(s, q) \leq a_*(s, q) < b_*(s, q)$ , for each  $0 < q < s$  fixed.

are satisfied, for  $0 < a'(s, q) \vee q < a < b < s$  and every  $i = 1, 2$ . Hence, we may conclude that the functions  $\Xi_i(a; s, q)$ , for  $i = 1, 2$ , increase on the interval  $(a'(s, q), rs/(K\delta(s, q)))$  with  $\Xi_1(a'(s, q); s, q) > -\infty$ ,  $\Xi_2(0+; s, q) = -\infty$  and  $\Xi_i(rs/(K\delta(s, q)); s, q) > 0$ , for  $i = 1, 2$ , so that the ranges of their values are given by the intervals  $(\Xi_1(a'(s, q); s, q), \Xi_1(rs/(K\delta(s, q)); s, q))$  and  $(-\infty, \Xi_2(rs/(K\delta(s, q)); s, q))$ , respectively. Also, the functions  $\Upsilon_i(b; s, q)$ , for  $i = 1, 2$ , decrease on the interval  $(a'(s, q) \vee rq/(L\delta(s, q)), \infty)$  with  $\Upsilon_i(a'(s, q) \vee rq/(L\delta(s, q)); s, q) > 0$ , for  $i = 1, 2$ , and  $\Upsilon_1(\infty; s, q) = -\infty$ ,  $\Upsilon_2(\infty; s, q) = 0$ , so that the ranges of their values are given by the intervals  $(-\infty, \Upsilon_1(a'(s, q) \vee rq/(L\delta(s, q)); s, q))$  and  $(0, \Upsilon_2(a'(s, q) \vee rq/(L\delta(s, q)); s, q))$ , respectively.

We now observe that, when  $\Xi_i(rs/(K\delta(s, q)); s, q) \leq \Upsilon_i(a'(s, q) \vee rq/(L\delta(s, q)); s, q)$  holds, one can determine some  $\widehat{b}_i(s, q) \geq a'(s, q) \vee rq/(L\delta(s, q))$  from the equation  $\Xi_i(rs/(K\delta(s, q)); s, q) = \Upsilon_i(\widehat{b}_i(s, q); s, q)$ , while when  $\Xi_i(rs/(K\delta(s, q)); s, q) \geq \Upsilon_i(a'(s, q) \vee rq/(L\delta(s, q)); s, q)$  holds, one can determine some  $a'(s, q) \leq \widehat{a}_i(s, q) \leq rs/(K\delta(s, q))$  from the equation  $\Xi_i(\widehat{a}_i(s, q); s, q) = \Upsilon_i(a'(s, q) \vee rq/(L\delta(s, q)); s, q)$ , for each  $0 < q < s$  and every  $i = 1, 2$ . Hence, it follows from the equations in (3.37) that, for each  $a \in (a'(s, q), \widehat{a}_1(s, q) \wedge rs/(K\delta(s, q)))$ , there exists a unique number  $b \in (a'(s, q) \vee rq/(L\delta(s, q)) \vee \widehat{b}_1(s, q), \bar{b}(s, q))$ , while, for each  $a \in (a'(s, q) \vee \underline{a}(s, q), \widehat{a}_2(s, q) \wedge rs/(K\delta(s, q)))$ , there exists a unique number  $b \in (a'(s, q) \vee rq/(L\delta(s, q)) \vee \widehat{b}_2(s, q), \infty)$ , for each  $0 < q < s$  fixed. (Note that the values  $\underline{a}(s, q)$  and  $\bar{b}(s, q)$ , for all  $0 < q < s$ , are specified in Theorem 5.1 below.) In other words, we may conclude that the first and second equations in (3.37) uniquely determine the function  $b_1(a)$  on  $(a'(s, q), \widehat{a}_1(s, q) \wedge rs/(K\delta(s, q)))$  with the range  $(a'(s, q) \vee rq/(L\delta(s, q)) \vee \widehat{b}_1(s, q), \bar{b}(s, q))$  and the function  $b_2(a)$  on  $(a'(s, q) \vee \underline{a}(s, q), \widehat{a}_2(s, q) \wedge rs/(K\delta(s, q)))$  with the range  $(a'(s, q) \vee$

$rq/(L\delta(s, q)) \vee \widehat{b}_2(s, q), \infty)$ , for each  $0 < q < s$  fixed, respectively. These arguments imply that the expression  $a'(s, q) \vee rq/(L\delta(s, q)) \vee \widehat{b}_1(s, q) < b_1(0+) \equiv \bar{b}(s, q) < \infty \equiv b_2(\underline{a}(s, q))$  holds too. Moreover, the same arguments and assumptions directly yield that there exists exactly one intersection point with the coordinates  $a_*(s, q)$  and  $b_*(s, q)$  of the curves associated with the functions  $b_1(a)$  and  $b_2(a)$  on the interval  $a \in (a'(s, q) \vee \underline{a}(s, q), \widehat{a}_2(s, q) \wedge rs/(K\delta(s, q)))$  such that  $a'(s, q) \vee rq/(L\delta(s, q)) \vee \widehat{b}_1(s, q) < b_1(a_*(s, q)) \equiv b_*(s, q) \equiv b_2(a_*(s, q)) < \bar{b}(s, q)$  holds, for each  $0 < q < s$  (see Figure 3 above).

More precisely, let us assume that there exists at least two intersection points  $(a_*(s, q), b_*(s, q))$  and  $(\tilde{a}(s, q), \tilde{b}(s, q))$  of the curves  $b_1(a)$  and  $b_2(a)$  such that  $a'(s, q) \vee \underline{a}(s, q) < \tilde{a}(s, q) < a_*(s, q) \leq rs/(K\delta(s, q)) \wedge \widehat{a}_2(s, q)$  and  $a'(s, q) \vee rq/(L\delta(s, q)) \vee \widehat{b}_1(s, q) \leq \tilde{b}(s, q) < b_*(s, q) < \bar{b}(s, q)$  [or  $a'(s, q) \vee \underline{a}(s, q) < a_*(s, q) < \tilde{a}(s, q) \leq rs/(K\delta(s, q)) \wedge \widehat{a}_2(s, q)$  and  $a'(s, q) \vee rq/(L\delta(s, q)) \vee \widehat{b}_1(s, q) \leq b_*(s, q) < \tilde{b}(s, q) < \bar{b}(s, q)$ ] as well as  $b_2(a) > b_1(a)$ , for  $a \in (\tilde{a}(s, q), a_*(s, q))$ , and any  $0 < q < s$  fixed. Observe that, by virtue of assumptions made above and according to the implicit function theorem, it follows from the representations in (3.39)-(3.40) that the expressions:

$$b'_i(a) = \frac{\Xi'_i(a)}{\Upsilon'_i(b)} = \frac{K(a - rs/(K\delta(s, q)))}{L(b - rq/(L\delta(s, q)))} \left(\frac{b}{a}\right)^{\gamma_{3-i}(s, q)+1} < 0 \quad (3.42)$$

hold, for every  $i = 1, 2$ , for all  $a \in (\tilde{a}(s, q), a_*(s, q))$  and  $b \in (\tilde{b}(s, q), b_*(s, q))$ , from where it directly follows that the inequality:

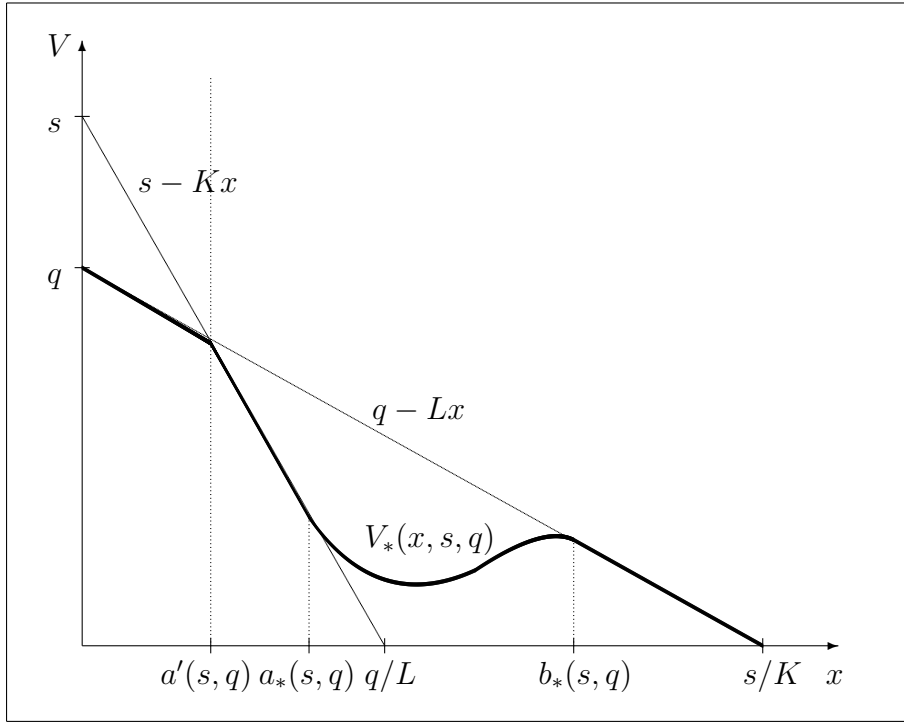
$$\frac{b'_2(a)}{b'_1(a)} = \left(\frac{b}{a}\right)^{\gamma_1(s, q) - \gamma_2(s, q)} > 1 \quad (3.43)$$

is satisfied, for all  $a \in (\tilde{a}(s, q), a_*(s, q))$ . Since the derivatives  $b'_i(a)$ , for  $i = 1, 2$ , from (3.42) are continuous functions on  $(\tilde{a}(s, q), a_*(s, q))$ , we may conclude that there exists an open interval  $(\tilde{a}(s, q) - \varepsilon, \tilde{a}(s, q) + \varepsilon)$ , for some relatively small  $\varepsilon > 0$ , such that the inequality  $b'_2(a) > b'_1(a)$  holds, so that the inequality  $b_2(a) > b_1(a)$  should hold, for  $a \in (\tilde{a}(s, q) - \varepsilon, \tilde{a}(s, q) + \varepsilon)$ , too. However, the latter fact contradicts the assumption that the equality  $b_1(\tilde{a}(s, q)) = b_2(\tilde{a}(s, q))$  holds, which means that the curves  $b_1(a)$  and  $b_2(a)$  may have only one intersection point, and thus, it completes the proof of the claim.

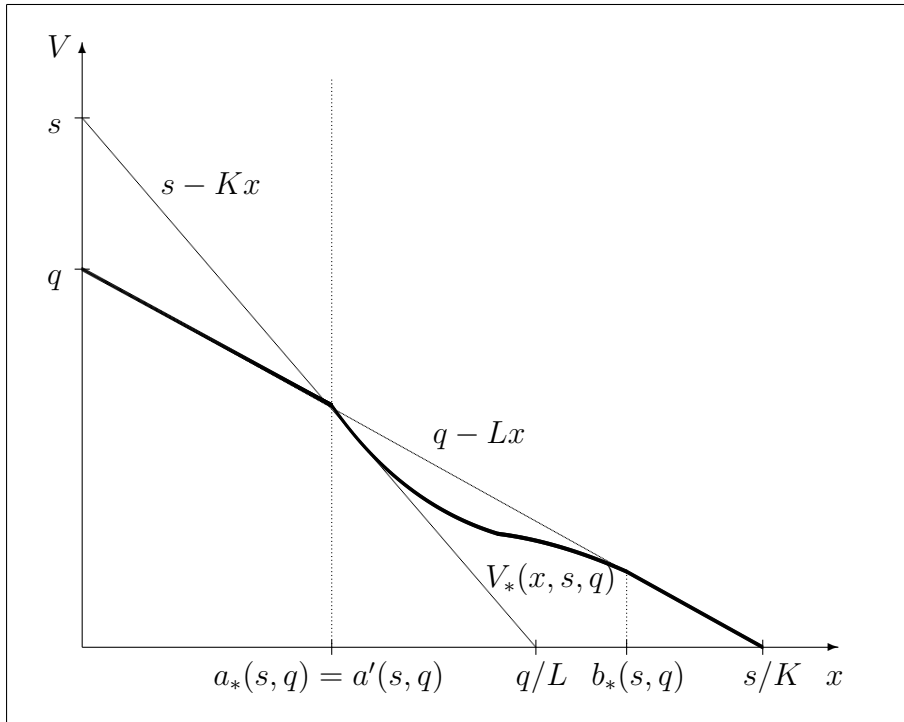
However, we recall that the functions  $a_*(s, q)$  and  $b_*(s, q)$  determined by means of the arguments above provide the candidate optimal stopping boundaries, whenever the inequalities  $a'(s, q) \leq a_*(s, q) < b_*(s, q)$  hold with  $a'(s, q) = (s - q)/(K - L)$ , for  $0 < q < s$ . Otherwise, the left-hand candidate stopping boundary  $a_*(s, q)$  coincides with  $a'(s, q)$ , while the right-hand candidate stopping boundary  $b_*(s, q) > a'(s, q)$  represents the largest root (or maximal solution) of the arithmetic equation in (3.22), which takes the form:

$$\begin{aligned} & \sum_{i=1}^2 (-1)^i \left( \frac{\gamma_{3-i}(s, q)q}{a'(s, q)} + (1 - \gamma_{3-i}(s, q)) L \frac{b(s, q)}{a'(s, q)} \right) \left( \frac{b(s, q)}{a'(s, q)} \right)^{-\gamma_i(s, q)} \\ & = (\gamma_1(s, q) - \gamma_2(s, q)) (s/a'(s, q) - K) \end{aligned} \quad (3.44)$$

for  $0 < q < s$ . We finally note that, in the case in which neither the system of arithmetic equations in (3.37) with (3.38) nor the equation in (3.44) admit solutions, for which either



**Figure 4.** A computer drawing of the value function  $V_*(x, s, q)$  and optimal exercise boundaries  $a'(s, q) < a_*(s, q) < b_*(s, q)$ , for each  $0 < q < s$  fixed.



**Figure 5.** A computer drawing of the value function  $V_*(x, s, q)$  and optimal exercise boundaries  $a'(s, q) = a_*(s, q) < b_*(s, q)$ , for each  $0 < q < s$  fixed.

the inequalities  $a'(s, q) \vee \underline{a}(s, q) \leq a_*(s, q) < b_*(s, q)$  or  $a'(s, q) < b_*(s, q) \leq \bar{b}(s, q)$  hold, respectively, we may conclude that the both candidate stopping boundaries  $a_*(s, q)$  and  $b_*(s, q)$  should coincide with  $a'(s, q)$ , for  $0 < q < s$ .

## 4. Main results and proofs

In this section, being based on the facts proved above, we formulate and prove the main result of the paper concerning the three-dimensional optimal stopping zero-sum game of (2.3) in the model from (1.2)-(1.3) and (1.4).

**Theorem 4.1** *Let the process  $(X, S, Q)$  be defined in (1.2)-(1.3) and (1.4), where  $r > 0$  is a constant, and  $\delta(s, q) > 0$  and  $\sigma(s, q) > 0$  are continuously differentiable bounded functions on  $[0, \infty]^2$ . Assume that the function  $\delta(s, q)$  is increasing in the both variables  $s$  and  $q$  on  $[0, \infty]^2$ . Then, the value function in (2.3) of the perpetual floating-strike lookback game option with some  $0 < L < K \leq 1$  given and fixed takes the form:*

$$V_*(x, s, q) = \begin{cases} V(x, s, q; a_*(s, q), b_*(s, q)), & \text{if } 0 < q \leq a_*(s, q) < x < b_*(s, q) \leq s \\ V(x, s, q; a_*(s, q)), & \text{if } 0 < q \leq a_*(s, q) < x \leq s < b_*(s, q) \\ V(x, s, q; b_*(s, q)), & \text{if } 0 < a_*(s, q) < q \leq x < b_*(s, q) \leq s \\ s - Kx, & \text{if } 0 < q \leq x \leq a_*(s, q) < s \\ q - Lx, & \text{if } 0 < q < b_*(s, q) \leq x \leq s \end{cases} \quad (4.1)$$

and the optimal exercise times forming a Nash equilibrium in the game are given by:

$$\tau_* = \inf \{t \geq 0 \mid X_t \leq a_*(S_t, Q_t)\} \quad \text{and} \quad \zeta_* = \inf \{t \geq 0 \mid X_t \geq b_*(S_t, Q_t)\} \quad (4.2)$$

where the stopping boundaries satisfy the inequalities  $a'(s, q) \vee \underline{a}(s, q) \leq a_*(s, q) \leq rs/(K\delta(s, q)) \wedge s$  and  $a'(s, q) \vee q \vee rq/(L\delta(s, q)) \leq b_*(s, q) \leq \bar{b}(s, q)$  with  $a'(s, q) = (s - q)/(K - L)$  (as well as  $\underline{a}(s, q)$  and  $\bar{b}(s, q)$  determined in Theorem 5.1 below), and together with the candidate value functions are further specified as follows:

(i) the function  $V(x, s, q; a_*(s, q), b_*(s, q))$  is given by (3.13)-(3.14), while either the boundaries  $a_*(s, q)$  and  $b_*(s, q)$  provide a unique solution of the system of arithmetic equations in (3.21), whenever  $a'(s, q) \leq a_*(s, q) < b_*(s, q)$  with  $a'(s, q) = (s - q)/(K - L)$  (see Figure 4 above), or  $a_*(s, q) = a'(s, q)$  and  $b_*(s, q)$  provides the largest root of the arithmetic equation in (3.22), whenever  $b_*(s, q) > a'(s, q)$  (see Figure 5 above), or  $a_*(s, q) = b_*(s, q) = a'(s, q)$  otherwise, in the regions  $\tilde{R}_{2k-1}(a_*, b_*)$  and  $\hat{R}_{2l-1}(a_*, b_*)$  from (3.29) and (3.33), for  $k, l \in \mathbb{N}$ , for  $0 < q < s$ ;

(ii) the function  $V(x, s, q; a_*(s, q))$  is given by (3.15)-(3.16), while the boundary  $a_*(s, q)$  provides either a unique solution of the first-order nonlinear ordinary differential equation in (3.23) started at  $(s_{2k-1}^*(q), s_{2k-1}^*(q), q)$ , whenever  $a_*(s, q) \geq a'(s, q)$ , or coincides with  $a'(s, q)$  otherwise, in the regions  $\tilde{R}_{2k}(a_*)$  from (3.30), for  $k \in \mathbb{N}$ , for  $0 < q < s$ ;

(iii) the function  $V(x, s, q; b_*(s, q))$  is given by (3.17)-(3.18), while the boundary  $b_*(s, q)$  provides a unique solution of the first-order nonlinear ordinary differential equation in (3.26) started at  $(q_{2l-1}^*(s), s, q_{2l-1}^*(s))$ , in the regions  $\hat{R}_{2l}(b_i^*)$  from (3.34), for  $l \in \mathbb{N}$ , for  $0 < q < s$ .

Observe that we can put  $s = q = x$  to obtain the value of the original perpetual floating-strike lookback game option pricing problem of (1.1) from the value function of the optimal stopping problem of (2.3).

**Proof** In order to verify the assertion stated above, it remains for us to show that the function defined in (4.1) coincides with the value function in (2.3) and the stopping times  $\tau_*$  and  $\zeta_*$  in (4.2) form a Nash equilibrium with the boundaries  $a_*(s, q)$  and  $b_*(s, q)$  specified in the previous section. For this purpose, let us denote by  $V(x, s, q)$  the right-hand side of the expression in (4.1) associated with these boundaries  $a_*(s, q)$  and  $b_*(s, q)$ . Then, it follows from the straightforward calculations presented in the previous section that the function  $V(x, s, q)$  solves the system of (2.27)-(2.29), while the smooth-fit and normal-reflection conditions of (2.32)-(2.33) are satisfied in the appropriate regions  $\tilde{R}_{2k-1}(a_*, b_*)$ ,  $\tilde{R}_{2k}(a_*)$  from (3.29)-(3.30), for  $k \in \mathbb{N}$ , and  $\hat{R}_{2l-1}(a_*, b_*)$ ,  $\hat{R}_{2l}(b_*)$  from (3.33)-(3.34), for  $l \in \mathbb{N}$ , respectively. We also observe that the function  $V(x, s, q)$  is  $C^{2,1,1}$  on the closure  $\bar{C}_*$  of  $C_*$  from (2.23) and  $D_*$  from (2.24), by construction. Hence, taking into account the fact that the boundaries  $a_*(s, q)$  and  $b_*(s, q)$  are assumed to be continuously differentiable, for  $0 < q < s$ , by applying the change-of-variable formula from [52; Theorem 3.1] (see also [57; Chapter II, Section 3.5] for a summary of the related results and further references) to the process  $e^{-rt}V(X_t, S_t, Q_t)$ , we obtain:

$$\begin{aligned} e^{-rt}V(X_t, S_t, Q_t) &= V(x, s, q) + M_t \\ &+ \int_0^t e^{-ru} (\mathbb{L}V - rV)(X_u, S_u, Q_u) I(X_u \neq S_u, X_u \neq Q_u, X_u \neq a_*(S_u, Q_u), X_u \neq b_*(S_u, Q_u)) du \\ &+ \int_0^t e^{-ru} \partial_s V(X_u, S_u, Q_u) I(X_u = S_u) dS_u + \int_0^t e^{-ru} \partial_q V(X_u, S_u, Q_u) I(X_u = Q_u) dQ_u \end{aligned} \quad (4.3)$$

for all  $t \geq 0$ . Here, the process  $M = (M_t)_{t \geq 0}$  defined by:

$$M_t = \int_0^t e^{-ru} \partial_x V(X_u, S_u, Q_u) I(X_u \neq S_u, X_u \neq Q_u) \sigma(S_u, Q_u) X_u dB_u \quad (4.4)$$

is a continuous local martingale under the probability measure  $P_{x,s,q}$ . Note that, since the time spent by the process  $(X, S, Q)$  at the boundary surfaces  $\{(x, s, q) \in E' \mid x = a_*(s, q)\}$  and  $\{(x, s, q) \in E' \mid x = b_*(s, q)\}$  as well as at the planes  $d_1 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q \leq x = s\}$  and  $d_2 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q = x \leq s\}$  is of the Lebesgue measure zero (see, e.g. [10; Chapter II, Section 1]), the indicators in the second line of the formula of (4.3) as well as in the formula of (4.4) can be ignored. Moreover, since the process  $S$  increases only when the process  $(X, S, Q)$  is located on the plane  $d_1$ , while the process  $Q$  decreases only when the process  $(X, S, Q)$  is located on the plane  $d_2$ , the indicators in the third line of (4.3) can be set equal to one. Finally, taking into account the fact that the candidate value function  $V(x, s, q)$  satisfies the normal reflection conditions of (2.33) at the diagonals  $d_1$  and  $d_2$  in the regions  $\tilde{R}_{2k}(a_*)$ , for  $k \in \mathbb{N}$ , in (3.34) and  $\hat{R}_{2l}(b_*)$ , for  $l \in \mathbb{N}$ , we may conclude that the integrals in the third line of (4.3) are actually equal to zero.

By using straightforward calculations and the arguments from the previous section, it is verified that the inequality  $(\mathbb{L}V - rV)(x, s, q) \leq 0$  holds, for all  $a'(s, q) < x < b_*(s, q)$  such that  $x \neq a_*(s, q)$ , and the inequality  $(\mathbb{L}V - rV)(x, s, q) \geq 0$  holds, for all  $x > a_*(s, q)$  such that

$x \neq b_*(s, q)$ , as well as  $x \neq s$  and  $x \neq q$ . Moreover, we observe directly from the expressions in (3.13), (3.15), and (3.17) that the function  $V(x, s, q) - (s - Kx)$  increases in the variable  $x$  on the interval  $q \vee a_*(s, q) < x < b_*(s, q) \wedge s$  from 0 (when  $a_*(s, q) \geq q$ ) to the value  $q - s + (K - L)b_*(s, q)$  (when  $b_*(s, q) \leq s$ ), because the expression  $\partial_x V(x, s, q) + K$  for the first-order partial derivative in (3.19) is positive there, for  $(x, s, q) \in E'$ . We also note from the expressions in (3.13), (3.15), and (3.17) that the function  $V(x, s, q)$  is convex in the variable  $x$  in a left-hand neighborhood of  $q \vee a_*(s, q)$  and concave in a right-hand neighborhood of  $b_*(s, q) \wedge s$ , because its second-order partial derivative  $\partial_{xx} V(x, s, q)$  is positive in a left-hand neighborhood of  $q \vee a_*(s, q)$  and negative in a right-hand neighborhood of  $b_*(s, q) \wedge s$ , for  $(x, s, q) \in E'$ . Thus, we may conclude that the inequalities in (2.30) hold, which together with the conditions of (2.28)-(2.29) and (2.32) imply that the inequalities  $s - Kx \leq V(x, s, q) \leq q - Lx$  are satisfied, for all  $(x, s, q) \in E'$ , and  $0 < L < K \leq 1$  given and fixed. It therefore follows from the expression in (4.3) that the inequalities:

$$e^{-r(\tau \wedge \zeta_*)} (S_{\tau \wedge \zeta_*} - K X_{\tau \wedge \zeta_*}) \leq e^{-r(\tau \wedge \zeta_*)} V(X_{\tau \wedge \zeta_*}, S_{\tau \wedge \zeta_*}, Q_{\tau \wedge \zeta_*}) \leq V(x, s, q) + M_{\tau \wedge \zeta_*} \quad (4.5)$$

and

$$e^{-r(\tau_* \wedge \zeta)} (Q_{\tau_* \wedge \zeta} - L X_{\tau_* \wedge \zeta}) \geq e^{-r(\tau_* \wedge \zeta)} V(X_{\tau_* \wedge \zeta}, S_{\tau_* \wedge \zeta}, Q_{\tau_* \wedge \zeta}) \geq V(x, s, q) + M_{\tau_* \wedge \zeta} \quad (4.6)$$

hold, for any stopping times  $\tau$  and  $\zeta$  of the process  $(X, S, Q)$ .

Now, consider the localising sequence  $(\varkappa_n)_{n \in \mathbb{N}}$  for the local martingale  $M$  from (4.4) such that  $\varkappa_n = \inf\{t \geq 0 \mid |M_t| \geq n\}$ , for each  $n \in \mathbb{N}$ . Then, inserting  $\tau \wedge \varkappa_n$  and  $\zeta \wedge \varkappa_n$  instead of  $\tau$  and  $\zeta$  in (4.5) and (4.6) and taking the expectations with respect to the probability measure  $P_{x,s,q}$  in (4.5) and (4.6), by means of Doob's optional sampling theorem, we get:

$$\begin{aligned} E_{x,s,q} [e^{-r(\tau \wedge \zeta_* \wedge \varkappa_n)} ((S_{\tau \wedge \varkappa_n} - K X_{\tau \wedge \varkappa_n}) I(\tau \wedge \varkappa_n < \zeta_*) + (Q_{\zeta_*} - L X_{\zeta_*}) I(\zeta_* \leq \tau \wedge \varkappa_n))] \quad (4.7) \\ \leq E_{x,s,q} [e^{-r(\tau \wedge \zeta_* \wedge \varkappa_n)} V(X_{\tau \wedge \zeta_* \wedge \varkappa_n}, S_{\tau \wedge \zeta_* \wedge \varkappa_n}, Q_{\tau \wedge \zeta_* \wedge \varkappa_n})] \\ \leq V(x, s, q) + E_{x,s,q} [M_{\tau \wedge \zeta_* \wedge \varkappa_n}] = V(x, s, q) \end{aligned}$$

and

$$\begin{aligned} E_{x,s,q} [e^{-r(\tau_* \wedge \zeta \wedge \varkappa_n)} ((S_{\tau_*} - K X_{\tau_*}) I(\tau_* < \zeta \wedge \varkappa_n) + (Q_{\zeta \wedge \varkappa_n} - L X_{\zeta \wedge \varkappa_n}) I(\zeta \wedge \varkappa_n \leq \tau_*))] \quad (4.8) \\ \geq E_{x,s,q} [e^{-r(\tau_* \wedge \zeta \wedge \varkappa_n)} V(X_{\tau_* \wedge \zeta \wedge \varkappa_n}, S_{\tau_* \wedge \zeta \wedge \varkappa_n}, Q_{\tau_* \wedge \zeta \wedge \varkappa_n})] \\ \geq V(x, s, q) + E_{x,s,q} [M_{\tau_* \wedge \zeta \wedge \varkappa_n}] = V(x, s, q) \end{aligned}$$

for all  $(x, s, q) \in E'$  and each  $n \in \mathbb{N}$ . Observe that, taking into account the arguments from [62; pages 635-636], it follows from the structure of the stopping times  $\tau_*$  and  $\zeta_*$  in (4.2) that the property:

$$E_{x,s,q} \left[ \sup_{t \geq 0} e^{-r((\tau_* \wedge \zeta_*) \wedge t)} S_{(\tau_* \wedge \zeta_*) \wedge t} \right] = E_{x,s,q} \left[ \sup_{t \geq 0} e^{-r((\tau_* \wedge \zeta_*) \wedge t)} (X_{(\tau_* \wedge \zeta_*) \wedge t} \vee s) \right] < \infty \quad (4.9)$$

holds, and the variables  $e^{-r(\tau_* \wedge \zeta_*)} (S_{\tau_* \wedge \zeta_*} - K X_{\tau_* \wedge \zeta_*})$  and  $e^{-r(\tau_* \wedge \zeta_*)} (Q_{\tau_* \wedge \zeta_*} - L X_{\tau_* \wedge \zeta_*})$  are finite on the set  $\{\tau_* \wedge \zeta_* = \infty\}$  ( $P_{x,s,q}$ -a.s.) as well as  $P_{x,s,q}(\tau_* \wedge \zeta_* < \infty) = 1$ , for all  $(x, s, q) \in E'$ . Hence, letting  $n$  go to infinity and using Fatou's lemma, we obtain that the inequalities:

$$E_{x,s,q} [e^{-r(\tau \wedge \zeta_*)} ((S_{\tau} - K X_{\tau}) I(\tau < \zeta_*) + (Q_{\zeta_*} - L X_{\zeta_*}) I(\zeta_* \leq \tau))] \leq V(x, s, q) \quad (4.10)$$

and

$$E_{x,s,q} \left[ e^{-r(\tau_* \wedge \zeta)} \left( (S_{\tau_*} - K X_{\tau_*}) I(\tau_* < \zeta) + (Q_\zeta - L X_\zeta) I(\zeta \leq \tau_*) \right) \right] \geq V(x, s, q) \quad (4.11)$$

hold, for any stopping times  $\tau$  and  $\zeta$ , and all  $(x, s, q) \in E'$ . Therefore, using the fact that the function  $V(x, s, q)$  and the continuously differentiable boundaries  $a_*(s, q)$  and  $b_*(s, q)$  solve the second-order ordinary differential equation in (2.27) and satisfy the conditions of (2.28)-(2.29) and (2.32)-(2.33), inserting  $\tau_*$  in place of  $\tau$  and  $\zeta_*$  in place of  $\zeta$  into (4.10) and (4.11), we obtain that the equality:

$$E_{x,s,q} \left[ e^{-r(\tau_* \wedge \zeta_*)} \left( (S_{\tau_*} - K X_{\tau_*}) I(\tau_* < \zeta_*) + (Q_{\zeta_*} - L X_{\zeta_*}) I(\zeta_* \leq \tau_*) \right) \right] = V(x, s, q) \quad (4.12)$$

holds, so that the candidate function  $V(x, s, q)$  coincides with the value function  $V_*(x, s, q)$  of the optimal stopping game in (2.3), for all  $(x, s, q) \in E'$ , and the optimal stopping times  $\tau_*$  and  $\zeta_*$  form a Nash equilibrium of the zero-sum game. We finally recall from the results of Part (ii) of Subsection 2.2 above implied by standard comparison arguments applied to the value functions of the appropriate optimal stopping problems that the inequalities  $a'(s, q) \vee \underline{a}(s, q) \leq a_*(s, q) < rs/(K\delta(s, q)) \wedge s$  and  $a'(s, q) \vee q \vee rq/(L\delta(s, q)) < b_*(s, q) \leq \bar{b}(s, q)$  should hold, for  $0 < q < s$ , that completes the verification.  $\square$

## 5 Appendix

In this section, we derive closed-form expressions for the value functions and optimal stopping boundaries of some auxiliary optimal stopping problems, which provide the upper and lower bounds for the value function and optimal stopping boundaries of the original optimal stopping zero-sum game of (2.3).

**5.1 The optimal stopping and free-boundary problems.** In order to provide the upper and lower bounds for the value functions and optimal stopping boundaries in the optimal stopping game of (2.3) above, let us introduce the value functions  $\bar{V}(x, s, q)$  and  $\underline{V}(x, s, q)$  of the optimal stopping problems:

$$\bar{V}(x, s, q) = \sup_{\tau} E_{x,s,q} \left[ e^{-r\tau} (S_\tau - K X_\tau) \right] \quad \text{and} \quad \underline{V}(x, s, q) = \inf_{\zeta} E_{x,s,q} \left[ e^{-r\zeta} (Q_\zeta - L X_\zeta) \right] \quad (5.1)$$

for some given constants  $0 < K \leq 1$  and  $0 < L < 1$ , respectively, where the supremum and infimum are taken with respect to all stopping times  $\tau$  and  $\zeta$  of the process  $X$ . Observe that the expected rewards in (5.1) have the form of the one in (2.3) under the assumptions that  $\zeta = \infty$  and  $\tau = \infty$ , respectively. Note that, when the inequality  $K > 1$  would hold, by virtue of the structure of the process  $S$  in (1.4), we see that the payoff  $S_t - K X_t$  could be negative, for all  $t \geq 0$ , so that the perpetual American lookback put option from (5.1) should not have been exercised instantly in that case. Moreover, when the inequality  $L \geq 1$  would hold, by virtue of the structure of the process  $Q$  in (1.4), we see that the payoff  $Q_t - L X_t$  would always be negative, for all  $t \geq 0$ , so that the perpetual American lookback call option from (5.1) should be exercised instantly in that case. It is shown by means of arguments similar to the ones used



above (or using the easily shown convexity and concavity of the value functions  $\bar{V}(x, s, q)$  and  $\underline{V}(x, s, q)$  that the optimal stopping times in the problems of (5.1) have the form:

$$\bar{\tau} = \inf \{t \geq 0 \mid X_t \leq \underline{a}(S_t, Q_t)\} \quad \text{and} \quad \underline{\zeta} = \inf \{t \geq 0 \mid X_t \geq \bar{b}(S_t, Q_t)\} \quad (5.2)$$

with some functions  $\underline{a}(s, q)$  and  $\bar{b}(s, q)$ , for  $0 < q < s$ , to be determined (see [49] and [34] as well as [22; Section 3] and [21]).

Following the arguments from [49] and [34] (see also [22; Section 3] and [21]) extended to the considered three-dimensional model, we may conclude that the unknown value functions  $\bar{V}(x, s, q)$  and  $\underline{V}(x, s, q)$  from (5.1) and the unknown boundaries  $\underline{a}(s, q)$  and  $\bar{b}(s, q)$  from (5.2) should solve the equivalent free-boundary problems:

$$(\mathbb{L}V - rV)(x, s, q) = 0 \quad \text{for} \quad q \vee a(s, q) < x < s \quad \text{or} \quad (5.3)$$

$$(\mathbb{L}V - rV)(x, s, q) = 0 \quad \text{for} \quad q < x < b(s, q) \wedge s \quad (5.4)$$

$$V(x, s, q)|_{x=a(s, q)+} = s - K a(s, q), \quad V(x, s, q)|_{x=b(s, q)-} = q - L b(s, q) \quad (5.5)$$

$$\partial_x V(x, s, q)|_{x=a(s, q)+} = -K, \quad \partial_x V(x, s, q)|_{x=b(s, q)-} = -L \quad (5.6)$$

$$\partial_s V(x, s, q)|_{x=s-} = 0, \quad \partial_q V(x, s, q)|_{x=q+} = 0 \quad (5.7)$$

$$V(x, s, q) = s - K x \quad \text{for} \quad x < a(s, q), \quad V(x, s, q) = q - L x \quad \text{for} \quad x > b(s, q) \quad (5.8)$$

$$V(x, s, q) > s - K x \quad \text{for} \quad a(s, q) < x \leq s, \quad V(x, s, q) < q - L x \quad \text{for} \quad q \leq x < b(s, q) \quad (5.9)$$

$$(\mathbb{L}V - rV)(x, s, q) < 0 \quad \text{for} \quad x < a(s, q), \quad (\mathbb{L}V - rV)(x, s, q) > 0 \quad \text{for} \quad x > b(s, q) \quad (5.10)$$

where the conditions in (5.5)-(5.7) are satisfied, for each  $0 < q < s$ , respectively.

**5.2 Solutions to the free-boundary problems.** It follows from the arguments of [49] and [34] (see also [22; Section 3] and [21]) that the solution to the left-hand system in (5.3)+(5.5)-(5.10) has the form of (3.15) with (3.16), for  $0 < q \leq a(s, q) < x \leq s$ , while the boundary  $a(s, q)$  solves the first-order nonlinear ordinary differential equation in (3.23), for any  $q > 0$  fixed. We also see that the solution to the right-hand system in (5.4)+(5.5)-(5.10) has the form of (3.17) with (3.18),  $0 < q \leq x < b(s, q) \leq s$ , while the boundary  $b(s, q)$  solves the first-order nonlinear ordinary differential equation in (3.26), for any  $s > 0$  fixed.

We further consider the *maximal and minimal admissible* solutions of first-order nonlinear ordinary differential equations as the largest and smallest possible solutions  $\underline{a}(s, q)$  and  $\bar{b}(s, q)$  of the equations in (3.23) and (3.26) which satisfy the inequalities  $\underline{a}(s, q) < s$  and  $\bar{b}(s, q) > q$ , for all  $0 < q < s$ . By virtue of the classical results on the existence and uniqueness of solutions for first-order nonlinear ordinary differential equations, we may conclude that these equations admit (locally) unique solutions, in view of the facts that the right-hand sides in (3.23) and (3.26) are (locally) continuous in  $(s, q, a(s, q))$  and  $(s, q, b(s, q))$  and (locally) Lipschitz in  $a(s, q)$  and  $b(s, q)$ , for each  $(s, q)$  fixed (see also [50; Subsection 3.9] for similar arguments based on the analysis of other first-order nonlinear ordinary differential equations). Then, it is shown by means of technical arguments based on Picard's method of successive approximations that there exist unique solutions  $a(s, q)$  and  $b(s, q)$  to the equations in (3.23) and (3.26), started at some points  $(s_0, s_0, q)$  and  $(s, q_0, q_0)$  such that  $s_0 > 0$  and  $q_0 > 0$ , for each  $0 < q < s$  fixed (see also [32; Subsection 3.2] and [50; Example 4.4] for similar arguments based on the analysis of other first-order nonlinear ordinary differential equations).

Hence, in order to construct the appropriate functions  $\underline{a}(s, q)$  and  $\bar{b}(s, q)$  which satisfy the equations in (3.23) and (3.26) and stays strictly above or below the appropriate diagonal, for  $0 < q < s$ , respectively, we can follow the arguments from [56; Subsection 3.5] (among others) which are based on the construction of sequences of the so-called bad-good solutions which intersect the diagonals. For this purpose, for any positive sequences  $(s_k, q_k)_{k \in \mathbb{N}}$  and  $(s_l, q_l)_{l \in \mathbb{N}}$  such that  $s_k \uparrow \infty$  as  $k \rightarrow \infty$  and  $q_l \downarrow 0$  as  $l \rightarrow \infty$ , we can construct the sequence of solutions  $a_k(s, q)$ , for  $k \in \mathbb{N}$ , and  $b_l(s, q)$ , for  $l \in \mathbb{N}$ , to the equations (3.23) and (3.26) such that  $a_k(s_k, q_k) = s_k$  and  $b_l(s_l, q_l) = q_l$  holds, for each  $k, l \in \mathbb{N}$ . It follows from the structure of the equations in (3.23) and (3.26) that the properties  $\partial_s a_k(s_k, q_k) < 1$  and  $\partial_q b_l(s_l, q_l) > 1$  hold, for each  $k, l \in \mathbb{N}$  (see also [49; pages 979-982] for the analysis of solutions of the non-parametrised version of the first-order nonlinear differential equation of (3.23)). Observe that, by virtue of the uniqueness of solutions mentioned above, we know that each two curves  $s \mapsto a_k(s, q)$  and  $s \mapsto a_m(s, q)$  as well as  $q \mapsto b_l(s, q)$  and  $q \mapsto b_n(s, q)$  cannot intersect, for  $l, k, m, n \in \mathbb{N}$  such that  $k \neq m$  and  $l \neq n$ , and thus, we see that the sequence  $(a_k(s, q))_{k \in \mathbb{N}}$  is increasing and the sequence  $(b_l(s, q))_{l \in \mathbb{N}}$  is decreasing, so that the limits  $\underline{a}(s, q) = \lim_{k \rightarrow \infty} a_k(s, q)$  and  $\bar{b}(s, q) = \lim_{l \rightarrow \infty} b_l(s, q)$  exist, for each  $0 < q < s$ , respectively. We may therefore conclude that  $\underline{a}(s, q)$  and  $\bar{b}(s, q)$  provides the maximal and minimal solutions to the equations in (3.23) and (3.26) such that  $\underline{a}_k(s, q) < s$  holds, for each  $k \in \mathbb{N}$ , and  $\bar{b}_l(s, q) > q$  holds, for each  $l \in \mathbb{N}$ , for all  $0 < q < s$ .

Moreover, since the right-hand sides of the first-order nonlinear ordinary differential equations in (3.23) and (3.26) are (locally) Lipschitz in  $s$  and  $q$ , for each  $0 < q < s$ , respectively, one can deduce by means of Gronwall's inequality that the functions  $a_k(s, q)$  and  $b_l(s, q)$ , for each  $k, l \in \mathbb{N}$ , are continuous, so that the functions  $\underline{a}(s, q)$  and  $\bar{b}(s, q)$  are continuous too. The appropriate *maximal admissible* solutions of first-order nonlinear ordinary differential equations and the associated maximality principle for solutions of optimal stopping problems which is equivalent to the superharmonic characterisation of the payoff functions were established in [50] and further developed in [32], [49], [34], [20], [5], [35], [55]-[56], [31], [48], [44], [28]-[30], [61], and [22] among other subsequent papers (see also [57; Chapter I; Chapter V, Section 17] for other references).

**5.3 The results.** Summarising the facts shown above, we state the following result which can be proved by means of the same arguments as Theorem 4.1 above in combinations with the arguments from [21]. Here, we also denote by  $\underline{a}(s, q; 0)$  and  $\bar{b}(s, q; 1) [\equiv q]$ , for  $0 < q < s$ , the optimal stopping boundaries in the problems of (5.1), under  $K = 0$  and  $L = 1$ , respectively (see [62]-[64] for the original derivation and [20]-[22] for the related comparison arguments).

**Theorem 5.1** *Let the process  $(X, S, Q)$  be defined in (1.2) and (1.4), where  $r > 0$  is a constant, and  $\delta(s, q) > 0$  and  $\sigma(s, q) > 0$  are continuously differentiable bounded functions on  $[0, \infty]^2$ . Assume that the function  $\delta(s, q)$  is increasing in the both variables  $s$  and  $q$  on  $[0, \infty]^2$ . Then, the following assertions hold:*

(i) *the value function  $\bar{V}(x, s, q)$  of the left-hand optimal stopping problem in (5.1) with  $0 < K \leq 1$  takes the form:*

$$\bar{V}(x, s, q) = \begin{cases} V(x, s, q; \underline{a}(s, q)), & \text{if } 0 < \underline{a}(s, q) < x \leq s \\ s - Kx, & \text{if } 0 < x \leq \underline{a}(s, q) \end{cases} \quad (5.11)$$

and the optimal stopping time  $\bar{\tau}$  has the form of (5.2), where the function  $V(x, s, q; a(s, q))$  is given by (3.15)-(3.16), while the optimal stopping boundary  $\underline{a}(s, q) [\leq \underline{a}(s, q; 0)]$  provides the maximal solution of the first-order nonlinear ordinary differential equation in (3.23) staying below the diagonal  $d_1 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q \leq x = s\}$ , for  $0 < q < s$ ;

(ii) the value function  $\underline{V}(x, s, q)$  of the right-hand optimal stopping problem in (5.1) with  $0 < L < 1$  takes the form:

$$\underline{V}(x, s, q) = \begin{cases} V(x, s, q; \bar{b}(s, q)), & \text{if } 0 < q \leq x < \bar{b}(s, q) \\ q - Lx, & \text{if } x \geq \bar{b}(s, q) \end{cases} \quad (5.12)$$

and the optimal stopping time  $\underline{\zeta}$  has the form of (5.2), where the function  $V(x, s, q; b(s, q))$  is given by (3.17)-(3.18), while the optimal stopping boundary  $\bar{b}(s, q) [\leq \bar{b}(s, q; 1) \equiv q]$  provides the minimal solution of the first-order nonlinear ordinary differential equation in (3.26) staying above the diagonal  $d_2 = \{(x, s, q) \in \mathbb{R}^3 \mid 0 < q = x \leq s\}$ , for  $0 < q < s$ .

**Acknowledgments.** The author is grateful to the Editors and two anonymous Referees for their valuable suggestions which helped to essentially improve the motivation and presentation of the paper.

## References

- [1] ASMUSSEN, S., AVRAM, F. and PISTORIUS, M. (2003). Russian and American put options under exponential phase-type Lévy models. *Stochastic Processes and their Applications* **109** (79–111).
- [2] AVRAM, F., KYPRIANOU, A. E. and PISTORIUS, M. (2004). Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Annals of Applied Probability* **14**(1) (215–238).
- [3] BAURDOUX, E. J. and KYPRIANOU, A. E. (2004). Further calculations for Israeli options. *Stochastics* **76** (549–569).
- [4] BAURDOUX, E. J. and KYPRIANOU, A. E. (2008). The McKean stochastic game driven by a spectrally negative Lévy process. *Electronic Journal of Probability* **8** (173–197).
- [5] BAURDOUX, E. J. and KYPRIANOU, A. E. (2009). The Shepp-Shiryaev stochastic game driven by a spectrally negative Lévy process. *Theory of Probability and its Applications* **53** (481–499).
- [6] BAURDOUX, E. J., KYPRIANOU, A. E. and PARDO, J. C. (2011). The Gapeev-Kühn stochastic game driven by a spectrally positive Lévy process. *Stochastic Processes and their Applications* **121** (1266–1289).
- [7] BENSOUSSAN, A. and FRIEDMAN, A. (1974). Non-linear variational inequalities and differential games with stopping times. *Journal of Functional Analysis* **16** (305–352).
- [8] BENSOUSSAN, A. and FRIEDMAN, A. (1977). Nonzero-sum stochastic differential games with stopping times and free-boundary problems. *Transactions of American Mathematical Society* **231** (275–327).
- [9] BISMUT, J. M. (1977). Sur un problème de Dynkin. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **39** (31–53).
- [10] BORODIN, A. N. and SALMINEN, P. (2002). *Handbook of Brownian Motion*. (Second Edition) Birkhäuser, Basel.
- [11] CVITANIĆ, J. and KARATZAS, I. (1996). Backward stochastic differential equations with reflection and Dynkin games. *Annals of Probability* **24** (2024–2056).

- [12] DETEMPLE, J. (2006). *American-Style Derivatives: Valuation and Computation*. Chapman and Hall/CRC, Boca Raton.
- [13] DUBINS, L., SHEPP, L. A. and SHIRYAEV, A. N. (1993). Optimal stopping rules and maximal inequalities for Bessel processes. *Theory of Probability and its Applications* **38**(2) (226–261).
- [14] EKSTRÖM, E. and PESKIR, G. (2008). Optimal stopping games for Markov processes. *SIAM Journal on Control and Optimization* **47**(2) (684–702).
- [15] EKSTRÖM, E. and VILLENEUVE, S. (2006). On the value of optimal stopping games. *Annals of Applied Probability* **16**(3) (1576–1596).
- [16] FRIEDMAN, A. (1973). Stochastic games and variational inequalities. *Archive for Rational Mechanics and Analysis* **51** (321–346).
- [17] FRIEDMAN, A. (1976). *Stochastic Differential Equations and Applications II*. Academic Press, New York.
- [18] GAPEEV, P. V. (2005). The spread option optimal stopping game. *Exotic Option Pricing and Advanced Levy Models*. Kyprianou, A., Schoutens, W. and Wilmott, P. (eds.) Wiley, Chichester (293–305).
- [19] GAPEEV, P. V. (2006). Discounted optimal stopping for maxima in diffusion models with finite horizon. *Electronic Journal of Probability* **11** (1031–1048).
- [20] GAPEEV, P. V. (2007). Discounted optimal stopping for maxima of some jump-diffusion processes. *Journal of Applied Probability* **44** (713–731).
- [21] GAPEEV, P. V. (2020). Optimal stopping problems for running minima with positive discounting rates. *Statistics and Probability Letters* **167** (108899).
- [22] GAPEEV, P. V., KORT, P. M. and LAVRUTICH, M. N. (2021). Discounted optimal stopping problems for maxima of geometric Brownian motions with switching payoffs. *Advances in Applied Probability* **53**(1) (189–219).
- [23] GAPEEV, P. V. and KÜHN, C. (2005). Perpetual convertible bonds in jump-diffusion models. *Statistics and Decisions* **23**(1) (15–31).
- [24] GAPEEV, P. V. and LERCHE, H. R. (2011). On the structure of discounted optimal stopping problems for one-dimensional diffusions. *Stochastics: An International Journal of Probability and Stochastic Processes* **83** (537–554).
- [25] GAPEEV, P. V. and LI, L. (2021). Optimal stopping problems for maxima and minima in models with asymmetric information. Published online first in *Stochastics: An International Journal of Probability and Stochastic Processes* (30 pp).
- [26] GAPEEV, P. V. and LI, L. (2021). Perpetual American defaultable standard and lookback options in models with incomplete information. *Submitted* (30 pp.).
- [27] GAPEEV, P. V. and AL MOTAIRI, H. (2021). Discounted optimal stopping problems in first-passage time models with random thresholds. To appear in *Journal of Applied Probability* (20 pp).
- [28] GAPEEV, P. V. and RODOSTHENOUS, N. (2014). Optimal stopping problems in diffusion-type models with running maxima and drawdowns. *Journal of Applied Probability* **51**(3) (799–817).
- [29] GAPEEV, P. V. and RODOSTHENOUS, N. (2016). On the drawdowns and drawups in diffusion-type models with running maxima and minima. *Journal of Mathematical Analysis and Applications* **434**(1) (413–431).
- [30] GAPEEV, P. V. and RODOSTHENOUS, N. (2016). Perpetual American options in diffusion-type models with running maxima and drawdowns. *Stochastic Processes and their Applications* **126**(7) (2038–2061).
- [31] GLOVER, K., HULLEY, H. and PESKIR, G. (2013). Three-dimensional Brownian motion and the golden ratio rule. *Annals of Applied Probability* **23** (895–922).

- [32] GRAVERSEN, S. E. and PESKIR, G. (1998). Optimal stopping and maximal inequalities for geometric Brownian motion. *Journal of Applied Probability* **35**(4) (856–872).
- [33] GRAVERSEN, S. E. and PESKIR, G. (1998). Optimal stopping and maximal inequalities for linear diffusions. *Journal of Theoretical Probability* **11** (259–277).
- [34] GUO, X. and SHEPP, L. A. (2001). Some optimal stopping problems with nontrivial boundaries for pricing exotic options. *Journal of Applied Probability* **38**(3) (647–658).
- [35] GUO, X. and ZERVOS, M. (2010).  $\pi$  options. *Stochastic Processes and their Applications* **120**(7) (1033–1059).
- [36] HENRY-LABORDÈRE, P. (2009). Calibration of local stochastic volatility models to market smiles: A Monte-Carlo approach. *Risk Magazine* **22**(10).
- [37] KALLSEN, J. and KÜHN, C. (2004). Pricing derivatives of American and game type in incomplete markets. *Finance and Stochastics* **8**(2) (261–284).
- [38] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*. (Second Edition) Springer, New York.
- [39] KIFER, Y. (2000). Game options. *Finance and Stochastics* **4** (443–463).
- [40] KITAPBAYEV, Y. (2014). On the lookback option with fixed strike. *Stochastics* **86**(3) (510–526).
- [41] KRYLOV, N. V. (1971). Control of Markov processes and W-spaces. *Izvestija* **5** (233–266).
- [42] KÜHN, C. and KYPRIANOU, A. E. (2007). Callable puts as composite exotic options. *Mathematical Finance* **17**(4) (487–502).
- [43] KYPRIANOU, A. E. (2004). Some calculations for Israeli options. *Finance and Stochastics* **8**(1) (73–86).
- [44] KYPRIANOU, A. E. and OTT, C. (2014). A capped optimal stopping problem for the maximum process. *Acta Applicandae Mathematicae* **129** (147–174).
- [45] LEPELTIER, J. P. and MAINGUENAU, M. A. (1984). Le jeu de Dynkin en théorie générale sans l’hypothèse de Mokobodski. *Stochastics* **13** (25–44).
- [46] LIPTSER, R. S. and SHIRYAEV, A. N. (2001). *Statistics of Random Processes I*. (Second Edition, First Edition 1977) Springer, Berlin.
- [47] NEVEU, J. (1975). *Discrete-Parameter Martingales*. North-Holland, Amsterdam.
- [48] OTT, C. (2013). Optimal stopping problems for the maximum process with upper and lower caps. *Annals of Applied Probability* **23** (2327–2356).
- [49] PEDERSEN, J. L. (2000). Discounted optimal stopping problems for the maximum process. *Journal of Applied Probability* **37**(4) (972–983).
- [50] PESKIR, G. (1998). Optimal stopping of the maximum process: The maximality principle. *Annals of Probability* **26** (1614–1640).
- [51] PESKIR, G. (2005). The Russian option: finite horizon. *Finance and Stochastics* **9** (251–267).
- [52] PESKIR, G. (2007). A change-of-variable formula with local time on surfaces. *Séminaire de Probabilité, Lecture Notes in Mathematics* **1899** (69–96).
- [53] PESKIR, G. (2008). Optimal stopping games and Nash equilibrium. *Theory of Probability and its Applications* **53** (558–571).
- [54] PESKIR, G. (2012). A duality principle for the Legendre transform. *Journal of Convex Analysis* **19** (609–630).
- [55] PESKIR, G. (2012). Optimal detection of a hidden target: The median rule. *Stochastic Processes and their Applications* **122** (2249–2263).

- [56] PESKIR, G. (2014). Quickest detection of a hidden target and extremal surfaces. *Annals of Applied Probability* **24** (2340–2370).
- [57] PESKIR, G. and SHIRYAEV, A. N. (2006). *Optimal Stopping and Free-Boundary Problems*. Birkhäuser, Basel.
- [58] QIU, S. (2014). American strangle options. *Research Report* **22**(2014). Probability and Statistics Group, School of Mathematics, The University of Manchester.
- [59] REN, Y., MADAN, D., and QIAN, M. Q. (2007). Calibrating and pricing with embedded local volatility models. *Risk Magazine* **20**(9) 138.
- [60] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*. Springer, Berlin.
- [61] RODOSTHENOUS, N. and ZERVOS, M. (2017). Watermark options. *Finance and Stochastics* **21**(1) (157–186).
- [62] SHEPP, L. A. and SHIRYAEV, A. N. (1993). The Russian option: reduced regret. *Annals of Applied Probability* **3**(3) (631–640).
- [63] SHEPP, L. A. and SHIRYAEV, A. N. (1994). A new look at the pricing of Russian options. *Theory Probability and its Applications* **39**(1) (103–119).
- [64] SHEPP, L. A. and SHIRYAEV, A. N. (1996). A dual Russian option for selling short. In *Probability Theory and Mathematical Statistics: Lectures presented at the semester held in St. Peterburg, Russia, March 2 – April 23, 1993*. Ibragimov, I. A. et al. eds.
- [65] SHIRYAEV, A. N. (1978). *Optimal Stopping Rules*. Springer, Berlin.
- [66] SHIRYAEV, A. N. (1999). *Essentials of Stochastic Finance*. World Scientific, Singapore.
- [67] STETTNER, L. (1982). On a general zero-sum stochastic game with optimal stopping. *Probability and Mathematical Statistics* **3**(1) (103–112).